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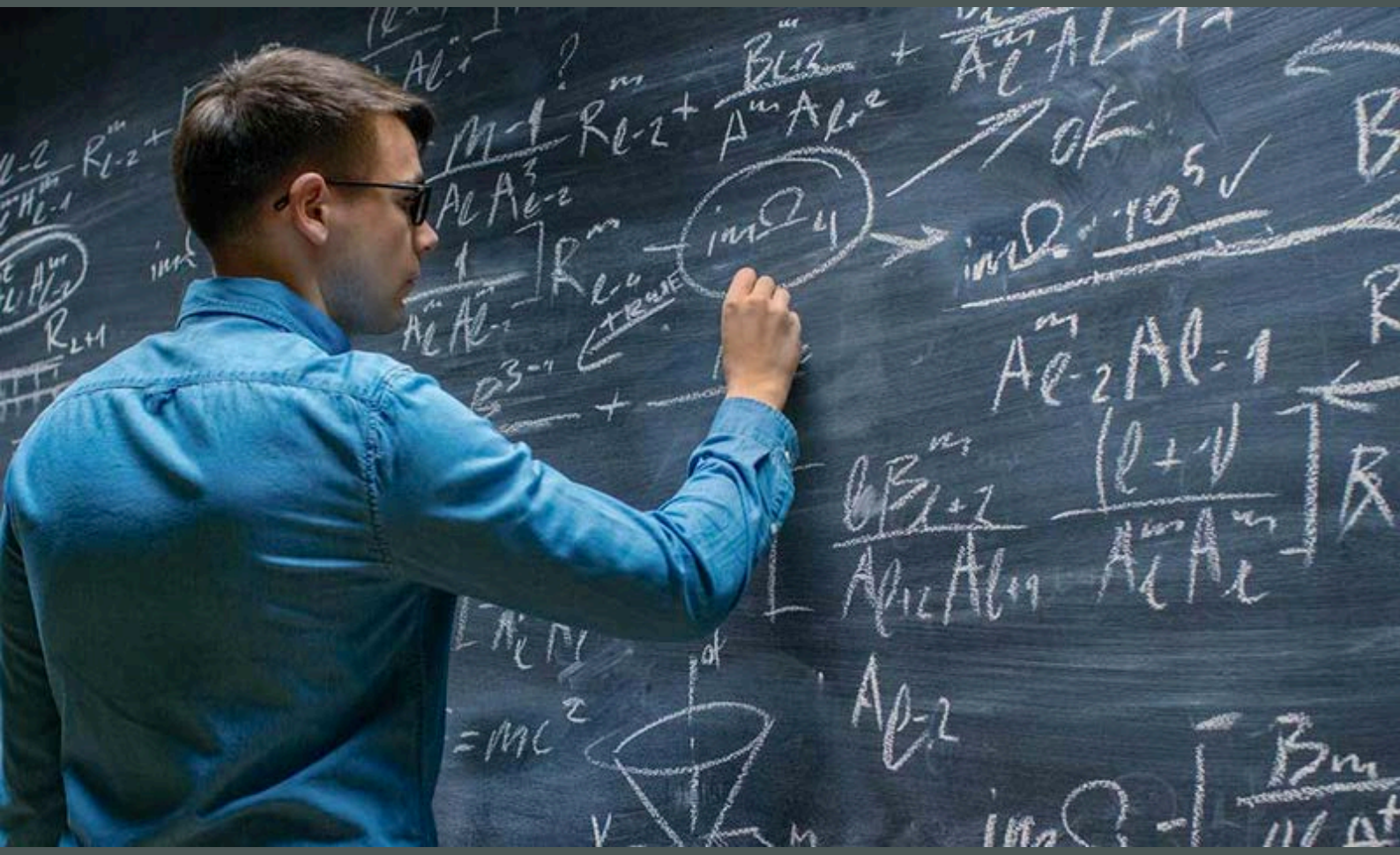

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PARTIAL DIFFERENTIAL EQUATIONS

A differential equation which involves partial derivatives is called a partial differential equations.

Such an equation involves one dependent variable and more then one independent variables.

Examples:

$$\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = z$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \text{ etc.}$$

x and y will, usually, be taken as the independent variables and z, the dependent variable so that z = f(x, y) and we shall employ the following notation:

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial^2 z}{\partial x^2} = r,\ \frac{\partial^2 z}{\partial x \partial y} = s,\ \frac{\partial^2 z}{\partial y^2} = t.$$

FORMATION OF PARTIAL EQUATIONS

Example

Derive a partial differential equation (by eliminating the constants) from the equation.

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad (1)$$

Differentiating (1) partially with respect to x and y, we get

$$2\frac{\partial z}{x} = \frac{2x}{a^2}$$

or $$\frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

and $$2\frac{\partial z}{\partial y} = \frac{2y}{b^2}$$

or $$\frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{p}{y}$$

Substituting these values of $\frac{1}{a^2}$ and $\frac{1}{b^2}$ in (1) , we get

$$2z = xp + yq$$

as the desired partial differential equation of the first order.

Example

Form the partial differential equation (by eliminating the arbitrary functions) from

(i) $z = f(x^2 + y^2)$

(ii) $z = f_1(x + ct) + f_2(x - ct)$

Solution

(i) We have $z = f(x^2+y^2)$

Differentiating z partially with respect to x and y,

$$p = \frac{\partial z}{\partial x} = f'(x^2 + y^2).2x$$

$$q = \frac{\partial z}{\partial x} = f'(x^2 + y^2).2y$$

Division gives , $\frac{p}{q} = \frac{x}{y}$

Or $yp - xq = 0$

which is the required equation.

(ii) We have $z = f_1(x + ct)+f_2(x - ct)$. (1)

Differentiating z partially with respect to x an t,

$$\frac{\partial z}{\partial x} = f_1'(x + ct) + f_2'(x - ct).$$

$$\frac{\partial^2 z}{\partial^2 x} = f_1''(x + ct) + f_2''(x - ct) \quad (2)$$

$$\frac{\partial z}{\partial t} = cf_1'(x + ct) + cf_2'(x - ct)$$

$$\frac{\partial^2 z}{\partial^2 t} = cf_2''(x + ct) + cf_2''(x - ct) = c^2 \frac{\partial^2 z}{\partial x^2} \quad \text{(by 2)}$$

Thus the desired partial differential equation is.

$$\frac{\partial^2 z}{\partial^2 t} = c^2 \frac{\partial^2 z}{\partial x^2}$$

which is an equation of the second order and (1) is its solution.

Example

Form partial diff equation by eliminating arbitrary function

$$z = (x+y)\phi(x^2 - y^2)$$

Solution

Differentiating

$$z_x = 1.\phi + (x+y)2x.\phi' \qquad (1)$$

and $$z_y = 1.\phi + (x+y)(-2y)\phi' \qquad (2)$$

From (2) $$\phi' = \frac{\phi - z_y}{2y(x+y)} \qquad (3)$$

Substituting (3) in (2)

$$z_x = \phi + 2x(x+y).\left[\frac{\phi - z_y}{2y(x+y)}\right]$$

$$p = \phi + \frac{x}{y}(\phi - q)$$

$$p = \left(\frac{x+y}{y}\right)\phi - \frac{x}{y}q$$

From given equation in problem

$$\phi = \frac{z}{x+y}$$

Substituting ϕ

$$p = \frac{(x+y)}{y}.\frac{z}{x+y} - \frac{x}{y}q = \frac{z}{y} - \frac{x}{y}q$$

or $$yp + xq = z$$

Example

Form partial diff equation by eliminating arbitrary function.

$$xyz = f(x+y+z)$$

Solution

Differentiating wrt x and y

$$yz + xyz_x = 1.f' + f'.z_x \quad (1)$$

$$xz + xyz_y = 1.f' + f'.z_y \quad (2)$$

From (2) $$f' = \frac{xz + xyz_y}{1 + z_y} = \frac{xz + xyq}{1 + q} \quad (3)$$

Put (3) in (1) $$yz + xyp = (1+p)f' = (1+p)\left(\frac{xz + xyq}{1+q}\right)$$

$$(1+q)(yz + xyp) = (1+p)(xz + xyq)$$

or $$x(y-z)p + y(z-x)q = z(x-y)$$

Problem

Form partial diff eq from

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Answer: $2z = xp + yq$

Problem

Form partial diff eq from

$$z = (x^2 + a^2)(y^2 + b^2)$$

Answer: $4xyz = pq$

Example

Form partial equation from

$$z = f(x).g(y)$$

Solution

Differentiating w.r.t. x and y, we get

$$z_x = f'g.z_y = fg'$$

$$z_x.z_y = f'gfg' = fgf'g' = zf'g'$$

But $$z_{xy} = f'g'$$

so $z_x.z_y = z.z_{xy}$

or $pq = z.s$

Example

Form partial differential equation from

$$z = f(x+y).g(x-y)$$

Solution

$$p = z_x = f'.1.g + f.1.g' \quad (1)$$

$$q = z_y = f'.1.g + f.(-1)g' \quad (2)$$

$$r = z_{xx} = f''g + f'g' + f'g' + fg''$$

$$= f''g + 2f'g' + fg'' \quad (3)$$

$$t = z_{yy} = f''g + f'g'(-1) - f'g' - fg''(-1)$$

$$= f''g - 2f'g' + fg'' \quad (4)$$

$$s = z_{xy} = f''g + f'g'(-1) + f'g' + fg''(-1)$$

$$= f''g - fg'' \quad (5)$$

Adding (1) and (2)

$$f' = \frac{p+q}{2g} \quad (6)$$

Subtracting (2) from (1)

$$g' = \frac{p-q}{2f} \quad (7)$$

Adding (3) and (4)

$$r + t = 2(f''g + fg'')$$

From (5) $2s = 2(f''g - fg'')$

Adding these two equations

$$r + t + 2s = 4f''g \quad (8)$$

Subtracting $r + t - 2s = 4fg''$

Substituting (6), (7), (8), (9) in (3)

$$r = \left(\frac{r+t+2s}{4}\right) + 2.\frac{(p+q)}{2g}.\frac{(p-q)}{2f} + \left(\frac{r+t-2s}{4}\right)$$

$\therefore$ $$(r-t)z = (p+q)(p-q)$$

Example

Form a partial differential equation from

$$z = xf(ax+by) + g(ax+by)$$

Solution

$$z_x = f + xaf' + ag'$$

$$z_{xx} = af' + af' + axf'' a + a^2 g'' = a[2f' + axf'' + ag''] \quad (1)$$

$$z_y = bxf' + bg'$$

$$z_{yy} = b^2 xf'' + b^2 g'' = b^2[xf'' + g''] \quad (2)$$

$$z_{yx} = bf' + bxaf'' + bag'' = b[f' + a(xf'' + g'')] \quad (3)$$

Substituting (2) in (3)

$$\frac{z_{yx}}{b} = f' + a.\frac{z_{yy}}{b^2}$$

Solving $$f' = \frac{z_{yx}}{b} - \frac{a}{b^2} z_{yy} \quad (4)$$

Substituting (2) and (4) in (1)

$$z_{xx} = 2af' + a^2[xf'' + g''] = 2a.\left[\frac{z_{yx}}{b} - \frac{a}{b^2} z_{yy}\right] + a^2\left[\frac{z_{yy}}{b^2}\right]$$

$$b^2 z_{xx} + a^2 z_{yy} = 2abz_{xy}$$

$$b^2 r + a^2 t = 2abs$$

Problem

Form partial diff equation from

$$z = f(x^2 - y^2)$$

Answer: $yp + xq = 0$

Problem

Form partial eq from

$$z = f(\sin x + \cos y)$$

Answer: $p\sin y + q\cos x = 0$

Example

Form partial diff equation by eliminating arbitrary function from

$$xyz = f(x + y + z)$$

Solution

Put u = x + y + z, v = xyz so that the given equation may be written as F(u, v) = 0.

Differentiating wrt x and y, we get

$$F_u(1 + z_x) + F_v(yz + xyz_x) = 0$$

$$F_u(1 + z_y) + F_v(xz + xyz_y) = 0$$

Eliminating F_u, F_v, we have

$$\begin{vmatrix} 1+p & yz+xyp \\ 1+q & xz+xyq \end{vmatrix}$$

$$(xz + xyq)(1 + p) - (1 + q)(yz + xyp) = 0$$

or $$x(z - y)p + (x - z)q + z(x - y) = 0$$

Example

Form partial diff equation from

$$F(x^2 + y^2 + z^2, z^2 - 2xy) = 0$$

Solution

Let $u = x^2 + y^2 + z^2; v = z^2 - 2xy$

Differentiating F partially w.r.t. x, we get

$$\frac{\partial F}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial F}{\partial u}.\frac{\partial u}{\partial z}.\frac{\partial z}{\partial x} + \frac{\partial F}{\partial v}.\frac{\partial v}{\partial x} + \frac{\partial F}{\partial v}.\frac{\partial v}{\partial z}.\frac{\partial z}{\partial x}$$

$$F_u.2x + F_u.2z.p + F_v(-2y) + F_v.2z.p = 0$$

Similarly $$\frac{\partial F}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial F}{\partial u}.\frac{\partial u}{\partial z}.\frac{\partial z}{\partial y} + \frac{\partial F}{\partial v}.\frac{\partial v}{\partial y} + \frac{\partial F}{\partial v}.\frac{\partial v}{\partial z}.\frac{\partial z}{\partial y}$$

$$F_u.2y + F_u.2z.q + F_v(-2x) + F_v.2z.q = 0$$

Solving $$\begin{vmatrix} x+zp & -y+zp \\ y+zq & -x+zq \end{vmatrix} = 0$$

$$(x+zp)(zq-x) - (zp-y)(y+zq) = 0$$

or $$xz(q-p) + yz(q-p) + (y^2 - x^2) = 0$$

or $$(x+y)[z(q-p) + (y-x)] = 0$$

Problem

Form partial diff equation from

$$F(xy + z^2, x + y + z) = 0$$

Answer: $(2z - x)p + (y - 2z)q = x - y$

Problem

Form partial diff equation from

$$F(x^2 + y^2, z - xy) = 0$$

Answer: $xq - yp = x^2 - y^2$

LINEAR EQUATIONS OF THE FIRST ORDER

A linear partial differential equation of the first order, commonly known as **Lagrange's linear equation** is of the form

$$Pp + Qq = R \qquad (1)$$

Where P,Q and R are functions of x, y, z.

Such an equation is obtained by eliminating an arbitrary function ϕ from $\phi(u, v) = 0$

Where u, v, are some functions of x, y, z.

To solve the equation Pp + Qq = R

Form the subsidiary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

⇩

Solve these simultaneous equations giving u = a and v = b as its solutions.

⇩

Write the solutions as f(u, v) = 0 or u = f(v)

You are advised to note carefully the methods used to get these solutions of the system $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, given below:

(a) *Method of Grouping*: We take two terms from the simultaneous equations, say $\frac{dx}{P}=\frac{dy}{Q}$ when z is absent or can be cancelled. Thus we have a different equation in x and y only which can be solved and we get one solution.

(b) *Method of Multipliers*: We use multipliers i, m, n (which are not always constants) and get $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{ldx+mdy+ndz}{lP+mQ+nR}$. These multipliers are often so chosen that $lP+mQ+nR=0$. Then $ldx+mdy+ndz=0$. Integrating this, we get one solution.

(c) Combination of methods (a) and (b)

Example

Solve $(mz-ny)\frac{\partial z}{\partial x}+(nx-lz)\frac{\partial z}{\partial y}=ly-mx.$

Solution

Here the subsidiary equations are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Using multipliers x, y, x, we get

Each fraction $=\frac{xdx+mdy+ndz}{0}$

$\therefore \quad xdx+ydy+zdz=0$

which on integration gives

$$x^2+y^2+z^2=0 \qquad (1)$$

Again using multipliers I,m,n, we have

Each fraction $\frac{ldx+mdy+ndz}{0}$

$\therefore \quad ldx+mdy+ndz=0$

which on integration gives

$$lx+my+nz=b \qquad (2)$$

Hence, from (1) and (2) the required solution is

$$x^2 + y^2 + z^2 = f(lx + my + nz)$$

Example

Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

Solution

Here the subsidiary equations are

$$\frac{dx}{x^2 + y^2 + z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

from the last two fractions, we have

$$\frac{dy}{y} = \frac{dz}{z}$$

which on integration gives

$$\log y = \log z + \log a \text{ or } \frac{y}{z} = a \qquad (1)$$

Using multipliers x, y, z, we have

$$\text{Each fraction} = \frac{xdx + ydy + zdz}{x\left(x^2 + y^2 + z^2\right)}$$

$$\therefore \qquad \frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

which on integration gives $\log(x^2+y^2+z^2) = \log z + \log b$

$$\text{or} \qquad \frac{x^2 + y^2 + z^2}{z} = b \qquad (2)$$

Hence , from (1) and (2) the required solution is

$$(x^2 + y^2 + z^2) = z\left(\frac{y}{z}\right).$$

Problem

Solve $y^2p - xyq = x(z - 2y)$

Answer: $\phi(x^2 + y^2, yz - y^2) = 0$

Example

Find general solution of $(y + z)p + (z + x)q = x + y$

Solution

The auxiliary equations are

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}$$

each term

$$\frac{dx-dy}{y-x}=\frac{dy-dz}{z-y}=\frac{dx+dy+dz}{2(x+y+z)} \qquad (1)$$

From the first two terms, we get

$$\frac{dx-dy}{y-x}=\frac{dy-dz}{z-y}$$

Integrating $\log(x-y)=\log(y-z)+\log a$

or $$\frac{x-y}{y-z}=a \qquad (2)$$

From last two terms in (1), we have

$$-\frac{dy-dz}{y-z}=\frac{dx+dy+dz}{2(x+y+z)}$$

Integrating $-\log(y-z)=\frac{1}{2}\log(x+y+z)-\log b$

or $$(y-z)\sqrt{x+y+z}=b \qquad (3)$$

From (2) and (3), the required solution is

$$\phi\left[(y-z)\sqrt{x+y+z},\frac{x-y}{y-z}\right]=0$$

Example

Solve $\mathrm{x}\frac{\partial \mathrm{z}}{\partial \mathrm{x}}+\mathrm{y}\frac{\partial \mathrm{z}}{\partial \mathrm{y}}+\mathrm{t}\frac{\partial \mathrm{z}}{\partial \mathrm{t}}=\mathrm{xyt}$

Solution

In the differential equation $P_p + Q_q = R$, there are two independent variables x and y. To solve this we have to find two independent solution satisfying the equations $\frac{\mathrm{dx}}{\mathrm{P}}=\frac{\mathrm{dy}}{\mathrm{q}}=\frac{\mathrm{dz}}{\mathrm{R}}$.

This procedure can be extended to first order first degree differential equations involving more than two independent variables. In the present case the number of independent variables is three i.e., x, y, t.

$\therefore$ The auxiliary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{t}=\frac{dz}{xyt}\ \frac{dx}{x}=\frac{dy}{y}$$

gives $\log x=\log y+\log a$ or $\frac{x}{y}=a$

Similarly $\frac{dy}{y}=\frac{dt}{t}$ gives $\frac{y}{t}=b$

Using the multipliers yt, st, sy, -3 we get

each term $=\frac{ytdx+xtdy+xydt-3dz}{ytx+xty+xyt-3xyt}$

Since ytx + sty + xyt - 3xyt = 0

$\therefore$ ytdx + xtdy + xydt - 3dz = 0

Integrating xyt-3z = C

Hence the general solution is $\phi=\left(\frac{x}{y},\frac{y}{t},xyt-3z\right)=0$

Problem

Solve $px(z-2y^2)=(z-qy)(z-y^2-2x^3)$

Answer: $f\left(\frac{y}{z},\frac{z}{x}-a^2,\frac{z^2}{x}+x^2\right)$

NON-LINEAR EQUATIONS OF THE FIRST ORDER

Those equations in which p and q occur other than in the first degree are called non-linear partial differential equations of the first order .

Form I. f(p, q) = 0 , i.e. equations containing p and q only

This type of equations does not contain x, y, z explicitly. For example $p^2+q^2=8$ is an equation of this type.

Method of Solution: We assume $z=ax+by+c$ as a solution.

Then $p = \frac{\partial z}{\partial x} = a, q = \frac{\partial z}{\partial y} = b$

$\therefore$ $z = ax + by + c$ is a solution where f(a, b) = 0 or b = ϕ(a).

or $z = ax + \phi(a)y + c$ is a complete solution.

Example

Solve $\sqrt{p} + \sqrt{q} = 1$.

Solution

The complete solution is

$$z = ax + by + c \qquad (1)$$

where $\sqrt{a} + \sqrt{b} = 1$ and $b = \left(1 - \sqrt{a}\right)^2$

$\therefore$ (1) becomes $z = ax + \left(1 - \sqrt{a}\right)^2 y + c$

which is the desired solution'

Problem

Solve $p^2 + q^2 = 8$

Answer: $z = ax + \sqrt{(8 - a^2 y)} + c$

Form II f(z, p, q) = 0 i.e. equations not containing x and y (eq. containing p, q, z only)

Assume $z = \phi(y + ax) = \phi(u)$ *as a solution.*

$$\Downarrow$$

$$p = \frac{\partial z}{\partial x} = \phi'(y + ax).a = a\phi'(u) = a\frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \phi'(y + ax) = \phi'(u) = \frac{\partial z}{\partial u}$$

$$\Downarrow$$

Substituting these values of p and q in the given equation, we get $f\left(z, a\frac{dz}{du}, \frac{dz}{du}\right) = 0$ *which is an ordinary diff. equation in u and z.*

Example

Solve $p(1 + q^2) = q(z - a)$

Solution

Let $z = \phi(y + ax) = \phi(u)$

Then $$p = \frac{\partial z}{\partial x} = a\phi'(y + ax) = a\phi'(u) = a\frac{dz}{du}$$

and $$q = \frac{\partial z}{\partial y} = \phi'(y + ax) = \phi'(u) = \frac{dz}{du}$$

Substituting the values of p and q in the given equation, we get

$$a\frac{dz}{du}\left[1 + \left(\frac{dz}{du}\right)^2\right] = \frac{dz}{du}(z - a)$$

Now either dz/du = 0 which gives z = c (a constant)

or $$a\left[1 + \left(\frac{dz}{du}\right)^2\right] = z - a$$

i.e. $$\left(\frac{dz}{du}\right)^2 = \frac{z - 2a}{a}$$

$\therefore$ $$\frac{dz}{du} = \frac{1}{\sqrt{a}}\sqrt{z - 2a}$$

Integrating $$\sqrt{a}\int \frac{dz}{\sqrt{z - 2a}} = u + b$$

or $$2\sqrt{a}\sqrt{z - 2a} = y + ax + b$$

or $4a(z - 2a) = (y + ax + b)^2$ is the required solution.

Problem

Solve $z = p^2 + q^2$

Answer: $4(a^2 + 1)z = (u + b)^2 = (y + ax + b)^2$

Form III f(x,p) = F(y,q), i.e. equations in which z is absent and the terms containing x and p can be separated from those containing y and q

As the trial solution assume that $f_1(x, p) = f_2(y, q) = a$, *say.*

⇩

let $p = F_1(x, a)$ *be the solution of* $f_1(x, p) = a$

and $q = F_2(y,a)$ *be that of* $f_2(y,q) = a$

$\Downarrow$

Since z is a function of x and y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$or \qquad dz = pdx + qdy$$

$\Downarrow$

Putting the values of p and q, we get

$$dz = F_1(x,a)dx + F_2(y,a)dy$$

$$Integrating \qquad z = \int F_1(x,a)dx + \int F_2(y)dy + b$$

Which is the desired complete solution containing two constants a and b.

Example

Solve $p - x^2 = q + y^2$

Solution

Let $p - x^2 = a \; and \; q + y^2 = a$

$\therefore \qquad p = a + x^2 \; and \; q = a - y^2$

Putting values of p and q in $dz = pdx + qdy$

$$dz = (a + x^2)dx + (a - y^2)dy$$

Solving, we get

$$z = ax + \frac{x^3}{3} + ay - \frac{y^2}{3} + b$$

Problem

Solve $yp + xq + pq = 0$

Answer: $z = \frac{1}{(a-1)}\frac{x^2}{2} - \frac{y^2}{2a} + b$

Form IV: z = px + qy + f(p, q) (Clairaut's form)

It can be easily verified that $z = ax + by + f(a,b)$ *is a solution, because*

$$\frac{\partial z}{\partial x} = p = a \ \ and \ \frac{\partial z}{\partial y} = q = b$$

Example

Solve $z = px + qy + 4p^2q^2$

Solution

The equation is of Clairaut's form.

hence the complete solution is

$$z = ax + by + 4a^2b^2$$

Charpit's Method

This is general method for finding the complete solution of non linear partial differential equations of first order.

Let $f(x, y, z, p, q) = 0$ (1)

be given differential equation.

If we can find another relation between p, q, x, y and z

say $F(x, y, z, p, q) = 0$ (2)

then we can solve (1) and (2) for p and q and substitute them in

$$dz = pdx + qdy \quad (3)$$

Solution of (3), if exists, is the complete solution of (1).

Equation (2) will be formed by using certain steps which would be clear from following example.

Example

Solve the equation $(p^2 + q^2)y = qz$

Solution

Here $f = (p^2 + q^2)y - qz$

$$\frac{\partial f}{\partial x} = 0$$
$$\frac{\partial f}{\partial y} = p^2 + q^2$$
$$\frac{\partial f}{\partial z} = -q$$
$$\frac{\partial f}{\partial p} = 2py$$
$$\frac{\partial f}{\partial q} = 2qy - z$$

The auxiliary system is

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

i.e. $$\frac{-dp}{-pq} = \frac{dq}{p^2} = \frac{dx}{-2py} = \frac{dy}{z - 2qy} = \frac{dz}{-qz}$$

From the first two ratios

$$pdq + qdp = 0$$

Integrating $p^2 + q^2 = c^2$ (1)

Solving (1) and the given equation $(p^2 + q^2)y = qz$

We get $q = \frac{c^2 y}{z}$

Substituting the value of q in (1), we have

$$p = \frac{c\sqrt{z^2 - c^2 y^2}}{z}$$

$\therefore$ $$dz = pdx + qdy = \frac{c}{z}\sqrt{z^2 - c^2 y^2}\,dx + \frac{c^2 y}{z}dy$$

or $$zdz - c^2 ydy = c\sqrt{z^2 - c^2 y^2}\,dx$$

or $$\frac{zdz - c^2 ydy}{\sqrt{z^2 - c^2 y^2}} = cdx$$

i.e. $$\frac{1}{2}\frac{d(z^2-c^2y^2)}{\sqrt{z^2-c^2y^2}} = cdx$$

Integrating $$\sqrt{z^2-c^2y^2} = cx+d$$

or $$z^2 = (cx+d)^2 + c^2y^2$$

Example

Solve $2xz - px^2 - 2qxy + pq = 0$

Solution

Here $f = 2xz - px^2 - 2qxy + pq$

$$\frac{\partial f}{\partial x} = 2z - 2px - 2qy$$

$\therefore$ $$\frac{\partial f}{\partial y} = 2qx$$

$$\frac{\partial f}{\partial z} = 2x$$

$$\frac{\partial f}{\partial p} = -x^2 + q$$

$$\frac{\partial f}{\partial q} = -2xy + p$$

The auxiliary equation is

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

or $$\frac{2p}{2z-2qy} = \frac{dq}{0} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dz}{px^2+2xyq}$$

From first two ratios dq = 0 or q = a.

Putting q = a in given equation, we get

$$p = \frac{2x(z-ay)}{x^2-a}$$

$\therefore$ $$dz = pdx + qdy$$

$$= \frac{2x(z-ay)dx}{x^2-a} + ady$$

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or $$\frac{dz-ady}{z-ay}=\frac{2xdx}{x^2-a}$$

i.e. $$\log(z-ay)=\log(x^2-a)+\log b=\log b(x^2-a)$$

or $z=ay+b(x^2-a)$ is complete solution.

Example

Solve $2z+p^2+qy+2y^2=0$

Solution

here $f=2z+p^2+qy+2y^2$

The auxiliary system in this case is

$$\frac{dp}{2p}=\frac{dq}{4y+3q}=\frac{dx}{-2p}=\frac{dy}{-y}=\frac{dz}{-(2p+qy)}$$

From first and third ratios

$$dp=-dx \text{ or } -p=x+a$$

Putting $p=-(x+a)$ in (1), we get

$$q=\frac{1}{y}\left[-2z-2z^2-(x+a)^2\right]$$

$\therefore$ $$dz=pdx+qdy=-(x+a)dx+\frac{1}{y}\left[-2z-2y^2-(x+a)^2\right]dy$$

or $$ydz=-y(x+a)dx-2zdy-2y^2dy-(x+a)^2dy$$

Multiplying both sides by 2y

$$2y^2dz+4yzdy=-2y^2(x+a)dx-2y(x+a)^2dy-4y^3dy$$

Integrating $$2zy^2=-[y^2(x+a)^2+y^4]+b$$

or $[y^2[(x+a)^2+2z+y^2]=b$ is the required solution.

Problem

Solve $z = pq$

Answer: $4az=(ax+y+b)^2$

Problem

Solve $p-3x^2=q^2-y$

Answer: $z=-(a-x)^3/3+x^3+ay-xy+b$

HOMOGENOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$\frac{\partial^2 z}{\partial x^n}+k_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+.....+k_n\frac{\partial^n z}{\partial y^n}=F(x,y) \qquad (1)$$

in which k's are constants, is called a *homogenous linear partial differential equation* of the nth order with constant coefficients. It is called homogeneous because all terms contain derivatives of the same order.

On writing, $\frac{\partial^r}{\partial x^r}=D^r$ and $\frac{\partial^r}{\partial y^r}=D'^r$,

(1) becomes $D^n+k_1D^{n-1}D'+.....+K_nD'^n=F(x,y)$

or briefly $f(D,D')z=F(x,y)$ (2)

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely: the *complementary function(C.F.)* and the *particular integral(P.I.).*

The complementary function is the compete solution of the equation f(D,D)z = 0, which must contain n arbitrary functions. The particular integral is the particular solution of the equation (2).

Rules for Finding the Complementary Function

Consider the equation

$$\frac{\partial^2 z}{\partial x^2}+k_1\frac{\partial^2 z}{\partial x\partial y}+k_2\frac{\partial^2 z}{\partial y^2}=0 \qquad (1)$$

which in symbolic form is

$$(D^2+k_1DD'+k_2D'^2)z=0 \qquad (2)$$

its symbolic operator equated to zero, i.e.

$$D^2+k_1DD'+k_2D'^2=0$$

is called the *auxiliary equation (A.E)*

Put $D/D'=m$

A.E. becomes $m^2+k_1m+k_2=0$

Rules for Finding the Particular Integral

Consider the equation

$$(D^2 + k_1 D D' + k_2 D'^2) z = 0$$

$$f(D,D') z = F(x, y)$$

$$\text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

Working Procedure to Solve the Equation

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial y^{n-1}} + \ldots\ldots\ldots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

Its symbolic form is

$$(D^n + k_1 D^{n-1} D' + \ldots\ldots k_n D'^n) z = F(x, y)$$

or briefly $f(D,D')z=(x, y)$.

Step I : To find the C.F

(i) Write the A.E.

i.e. $m^2+k_1m^{n-1}+\ldots..+ k_n=0$

and solve it for m.

(ii) Write the C.F. as follows

Roots of A.E.	**C.F.**
$m_1, m_2, m_3\ldots.$ (distinct roots)	$f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x) + \ldots$
$m_1, m_1, m_3\ldots.$ (two equal roots)	$f_1(y + m_1x) + xf_2(y + m_2x) + f_3(y + m_3x) + \ldots$
$m_1, m_2, m_1\ldots.$ (three equal roots)	$f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x) + \ldots$

Step II : To find the P.I.

From the symbolic form.

$$\text{P.I.} = \frac{1}{f(D, D')} F(x, y).$$

When F(x,y) = e^{ax+by}

$$P.I. = \frac{1}{f(D,D')}e^{ax+by} \qquad [\text{put } D = a \text{ and } D' = b]$$

When (x, y) = sin (mx + ny) or cos (mx + ny)

$$P.I. = \frac{1}{f\left(D^2, DD', D'^2\right)} \sin \text{ or } \cos (mx + ny)$$

[Put $D^2 = -m^2$, $D\,D' = -mn$, $D'^2 = -n^2$]

When F(x, y) = $x^m y^n$

$$P.I. = \frac{1}{f(D,D')}x^m y^n = \left[f\left(D,D'\right)\right]^{-1} x^m y^n$$

Expand $[f(D,D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term

When F(x, y) is any function of x and y

$$P.I. = \frac{1}{f(D,D')}F(x,y).$$

Resolve 1/f(D,D') into partial fractions considering f(D,D') as a function of D alone and operate each partial fraction on F(x, y) remembering that]

$$\frac{1}{D - mD'}F(x,y) = \int F\left(x,\ \ c - mx\right)dx$$

where c is replaced by y + mx after integration

Example

Solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^x + 2y$

Solution

Given equation in symbolic form is

$$(D^3 - 3D^2D' + 4D'^3)z = e^{x+2y}$$

Its A.E. is $m^3 - 3m^2 - 4 = 0$, whence m=-1,2,2.

$\therefore$ C.F. $= f_1(y-x) + f_2(x+2y) + xf_3(y+2x)$.

$$P.I. = \frac{1}{D^3 - 3D^2 + 4D'^3}e^{x+2y} \qquad [\text{Put } D=1, D'=2]$$

$$= \frac{1}{1^3 - 3.1^2 + 4.2^3}e^{x+2y} = \frac{e^{x+2y}}{27}$$

Hence the C.S. is

$$z = f_1(y-x) + f_2(y+2x) + xf_3(y+2x) + \frac{e^{x+2y}}{27}$$

Problem

Find P.I. of $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = e^{2x+3y}$

Answer: $\frac{e^{2x+3y}}{31}$

Example

Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$

Solution

Given equation in symbolic form is

$$(D^2 - DD')z = \cos x \cos 2y$$

Its I.E. is $m^2 - m = 0$ whence $m = 0,1$

$\therefore$ C.F. $= f_1(y) + f_2(y + x)$

$$P.I. = \frac{1}{D^2 - DD'}\cos x \cos 2y$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\left[\cos(x+2y) + \cos(x-2y)\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\cos(x+2y) + \frac{1}{D^2 - DD'}\cos(x-2y)\right]$$

Put $D^2 = -1$ and $DD' = -2$

Put $D^2 = -1$ and $DD' = 2$

$$= \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Hence the C.S. is

$$z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Problem

Solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x \partial y}-6\frac{\partial^2 z}{\partial y^2}=\cos(3x+y)$

Answer: $z=\phi_1(y-3x)+\phi_2(y+2x)-\frac{\cos(3x+y)}{6}$

Example

Solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x \partial y}-6\frac{\partial^2 z}{\partial y^2}=x+y$

Solution

The given equation is

$$(D^2 + DD' - 6D'^2)z = x + y$$

Here the A.E. is $m^2 + m - 6 = 0$ so that $m = 2, -3$

$$\text{C.F.} = \phi_1(y + 2x) + \phi_2(y - 3x)$$

$$\text{P.I.} = \frac{1}{D^2+DD'-6D'^2}(x+y)$$

$$=\frac{1}{D^2}\left[1+\left(\frac{D'}{D}-6\frac{D'^2}{D^2}\right)\right]^{-1}(x+y)$$

$$=\frac{1}{D^2}\left(1-\frac{D'}{D}\right)(x+y)$$

$$=\frac{1}{D^2}\left(x+y-\frac{1}{D}(0+1)\right)$$

$$=\frac{1}{D^2}(x+y-x)=\frac{1}{D^2}y=\frac{x^2y}{2}$$

$$\left[\because \frac{D'}{D}(x+y)=\frac{1}{D}\frac{\partial}{\partial y}(x+y)=\frac{1}{D}(1)=\int 1.dx = x\right]$$

$\therefore$ Complete solution is C.F. + P.I. $= \phi_1(y+2x)+\phi_2(y-3x)+\frac{x^2y}{2}$

Problem

Solve $\frac{\partial^2 z}{\partial x^2}+3\frac{\partial^2 z}{\partial x \partial y}+2\frac{\partial^2 z}{\partial y^2}=x+y$

Answer: $z = \phi_1(y-x) + \phi_2(y-2x) + \frac{x^2 y}{2} - \frac{x^3}{3}$

METHOD OF SEPARATION OF VARIABLES

This method helps us in breaking up the solution into a product of function each of which contains only one of the variables.

Example

Solve (by the method of separation of variables).

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

Solution

$$z = X(x)\,Y(y) \qquad (1)$$

where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X'' Y - 2 X' Y + XY' = 0$$

where $X' = \frac{dX}{dx}; Y' = \frac{dY}{dy}; X'' = \frac{d^2X}{dx^2}$ etc.

Separating the variables, we get

$$\frac{X''-2X'}{X} = -\frac{Y'}{Y} \qquad (2)$$

Since x and y are independent variables , therefore, (2) can only true if each side is equal to same constant , a (say).

$$\therefore \quad \frac{X''-2X'}{X} = a$$

i.e. $X''-2X'-aX = 0 \qquad (3)$

and $-\frac{Y'}{Y} = a$ i.e. $Y' = -aY \qquad (4)$

To solve the ordinary linear equation (3) the auxiliary equation is

$$m^2\text{-}2m\text{-}a=0, \quad \text{whence } m = 1 \pm \sqrt{(1+a)}$$

$\therefore$ The solution of (3) is $X = c_1 e^{\left[1+\sqrt{(1+a)}\right]x} + c_2 e^{\left[1-(1+a)x\right]}$

From (4), $Y'/Y = -a$ or $\log Y = -ay + c$

$$Y = e^{-ay+c} = c_3 e^{-ay} \qquad [\text{where } c_3 = e^c] \qquad (5)$$

Substituting the values of X and Y in (1), we get

$$z = \left[c_1 e^{[1+\sqrt{(1+a)}]x} + c_2 e^{\left[1-(1+a)x\right]} \right] c_3 e^{-ay}$$

$$z = \left[k_1 e^{[1+\sqrt{(1+a)}]x} + k_2 e^{\left[1-(1+a)x\right]} \right] e^{-ay}$$

which is the require complete solution.

Example

Using the method of separation of variables, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \text{ , } \textit{where } u(x,0) = 6e^{-3x}.$$

Solution

Before we solve the problem we should note that u is a function of x and t and therefore may be written as u(x, t). The value of u when t =0 is written as u(x,0) and its value when x = 0 is denoted by u(0,t).

Let u = XT be the solution.

Substituting in the given equation we have

$$X'T = 2XT' + XT$$

or $$(X' - X)T = 2XT'$$

or $$\frac{X' - X}{2X} = \frac{T'}{T} = k(\text{say})$$

$\therefore$ $$X' - X - 2kX = 0 \text{ or } X' - (1 + 2k)X = 0 \qquad (1)$$

and $$T' - kT = 0 \qquad (2)$$

From (1) $$\frac{X'}{X} = 1 + 2k$$

Solving $$\log X = (1 + 2k)x + c'$$

$$X = c_1 e^{(1+2k)x} \qquad [c_1 = c'] \qquad (3)$$

From (2) $$T'/T = k \qquad \text{or } \log T = kt + c''$$

$\therefore$ $$T = c_2 e^{kt} \qquad [c_2 = c''] \qquad (4)$$

From (3) and (4), we get

$$u = XT = c_1 c_2 e^{(1+2k)x} . e^{kt} = A . e^{(1+2k)x} . e^{kt} \qquad (5)$$

Now, $$u(x,0) = A . e^{(1+2k)x}$$

But $u(x,0) = 6e^{-3x}$ (given)

$\therefore$ $A.e^{(1+2k)x} = 6e^{-3x}$

Hence $A = 6$ and $1 + 2k = -3$ or $k = -2$

Putting these values in (5)

$u = 6e^{-3x}.e^{-2t} = 6e^{-(3x + 2t)}$ is the required solution.

Problem

Solve $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}, u(0, y) = 8e^{-3y}$

Answer: $u(x, y) = 8e^{-12x-3y}$

PARTIAL DIFFERENTIAL EQUATIONS OF ENGINEERING

In applied mathematics, the partial differential equations generally arise from the mathematical formulation of physical problems. The process of finding solutions of such and equation subject to given condition known as a boundary value problem.

We will study some of the boundary value problems related to the following equations of engineering :

- $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ This is known as the **wave equation** and occurs in the vibratory motion of strings and liquids.
- $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ This equation arise in one dimensional **heat flow** problems.
- $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial y^2} = 0$ This is the well known **Laplace equation** in tow dimensions and arise in the conduction of heat in a plate in steady state and in the irrigational fluid –flow.

The method of solution of such equations differ from that used in the case of ordinary differential equations. Here, in general starting with the method of separation of variables, we try to find those particular solutions which satisfy the given boundary conditions and the combination of such solutions gives the desired solution. Quite often a certain condition is not readily applicable. In such cases, the most general solution is written as the sum of the particular solution already found and the constants are determined using Fourier series, so as to satisfy the reaming condition.

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Example

Obtain the solution of the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ *using the method of separation of variables.*

Solution

Let $Y = XT$ (1)

be the solution of the given equation.

Substituting (1) in the equation we get

$XT'' = c^2X''T$

or $\frac{X''}{X} = \frac{1}{c^2} - \frac{T''}{T} = h(\text{say})$

Hence $X'' - hX = 0$ (2)

and $T'' - c^2hT = 0$ (3)

Now, three cases arise.

Case 1 (h > 0)

Let $h = k^2$

Then (2) and (3) become respectively

$X'' - k^2X = 0$ and $T'' - c^2k^2T = 0$

Solving these ordinary equations, we get

$X = c_1e^{kx} + c_2e^{-kx}$ and $T = c_3e^{ckt} + c_4e^{-ckt}$ (4)

Case II h = 0

Then (2) and (3) become $X'' = 0$ and $T'' = 0$

Solving we get

$X = c_5x + c_6$ and $T = c_7t + c_8$ (5)

Case III h < 0

Let $h = -k^2$

from (2) and (3) we get

$X'' + k^2X = 0$ and $T'' + c^2k^2T = 0$

Solving we get $X = c_9\cos kx + c_{10}\sin kx$ and $T = c_{11}\cos ckt + c_{12}\sin ckt$ (6)

Thus depending upon the value of h we have from (4), (5) and (6) the solution as

$y = (c_1e^{kx} + c_2e^{-kx})(c_3e^{ckt} + c_4e^{-ckt})$ (7)

$$y = (c_5 x + c_6)(c_7 t + c_8) \quad (8)$$

$$y = (c_9 \cos kx + c_{10} \sin kx)(c_{11} \cos ckt + c_{12} \sin ckt) \quad (9)$$

Since it is a problem on vibrating string every position of which is repeated as time passes, the solution should be a periodic function of t. Hence only (9) can be taken as the solution of the given equation.

Example

A string is attached and fastened to two points at a distance l apart. Motion is started by displacing the string in the form y = a sin(πx/l) from which it is released at a time t = 0. Show that the displacement of any point at distance x from one end at time t is given by

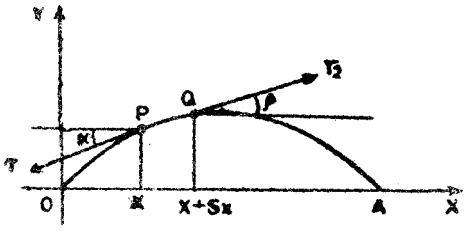
$$y(x,t) = a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}$$

Solution

The equation of the vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (1)$$

Let O and A be the points to which the ends are fastened so that OA = l.



Since the ends of the string are fixed, y = 0 when x = 0 and x = l, for all values of t.

Hence boundary conditions are

$$y(0, t) = 0 \text{ and } y(l, t) = 0 \quad (2)$$

The string is in the shape y = a sin(πx/l) and the rest when t = 0.

Hence the initial conditions are

$$y(x,0) = a \sin \frac{\pi x}{l}, \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad (3)$$

Now we have to solve (1) subject to the conditions (2) and (3).

As shown in previous example, the solution is of the form

$$y = (a_1 \cos kx + a_2 \sin kx)(a_3 \cos ckt + a_4 \sin ckt) \quad (4)$$

Since y(0,t) = 0 for all t (first boundary condition), therefore,

$$0 = a_1(a_3\cos ckt + a_4 \sin ckt)$$

Hence $a_1 = 0$.

Hence (4) reduces to

$$y(x,t) = a_2 \sin kx(a_3\cos ckt + a_4 \sin ckt) = \sin kx(b_1\cos ckt + b_2 \sin ckt)$$

[where $b_1 = a_2a_3$ and $b_2 = a_2a_1$]

From the second boundary condition y(l,t) = 0 for all t we get

$$0 = \sin kl\,(b_1\cos ckt + b_2 \sin ckt)$$

Hence sinkl = 0 or $kl = n\pi$ (n being an integer)

or $k = n\pi/l$

Hence $$y(x,t) = \sin\frac{n\pi x}{l}\left(b_1\cos\frac{cn\pi}{l}t + b_2\sin\frac{cn\pi}{l}t\right)$$

Now, $$\frac{\partial y}{\partial t} = \sin\frac{n\pi x}{l}\left(-b_1\frac{cn\pi}{l}\sin\frac{cn\pi t}{l} + b_2\frac{cn\pi}{l}\cos\frac{cn\pi t}{l}\right)$$

By second initial condition, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

Hence $0 = \sin\frac{n\pi x}{l}\left(b_2\frac{cn\pi}{l}\right)$ which gives $b_2 = 0$.

Hence $$y(x,t) = b_1\sin\frac{n\pi x}{l}\cos\frac{cn\pi t}{l} \qquad (5)$$

From this equation

$$y(x,0) = b_1\sin\frac{n\pi x}{l}$$

But $y(x,0) = a\sin\frac{\pi x}{l}$ (first initial condition)

Hence $$a\sin\frac{\pi x}{l} = b_1\sin\frac{n\pi x}{l}$$

This holds when $b_1 = a$ and $n = 1$.

Hence $y(x,t) = a\sin\frac{\pi x}{l}\cos\frac{c\pi t}{l}$ is the required solution.

Example

A tightly stretched string with fixed ends at x = 0 and x = l is initially in equilibrium position. It is set vibrating by giving to each of its points an initial velocity

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\frac{\pi x}{l}.$$

Find displacement y(x, t).

Solution

The equation of the vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

We have to solve this equation subject to the boundary conditions

$$y(0,t) = 0 \text{ and } y(l,t) = 0$$

and subject ti the initial conditions

$$y(x,0) = 0 \text{ and } \left(\frac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\frac{\pi x}{l}$$

Assuming the solution to be of the form y = XT, we get

$$XT'' = c^2X''T$$

or $$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -k^2(\text{cons tan t})$$

We have taken the constant as $-k^2$ so that solution is a periodic function of t.

Hence $$X'' + k^2X = 0 \text{ and } T'' + c^2k^2T = 0$$

Solving the differential equations, we get

$$X = c_1\cos kx + c_2\sin kx \text{ and } T = c_3\cos ckt + c_4\sin ckt$$

Hence the solution is of the form

$$y = (c_1\cos kx + c_2\sin kx)(c_3\cos ckt + c_4\sin ckt)$$

Now $$y(0,t) = 0 \text{ (first boundary condition)}$$

Hence $$0 = c_1(c_3\cos ckt + c_4\sin ckt)$$

This is true for all t when $c_1 = 0$.

Hence $$y(x,t) = c_2\sin kx(c_3\cos ckt + c_4\sin ckt) = \sin kx(a_1\cos ckt + a_2\sin ckt)$$

where $a_1 = c_2c_3$ and $a_2 = c_2c_4$

Also $y(l,t) = 0$ (second boundary condition)

Hence $0 = \sin kl\left(a_1 \cos ckt + a_2 \sin ckt\right)$ for all t.

This gives $kl = n\pi$ or $k = n\pi/l$. (n being an integer).

Hence $$y(x,t) = \sin\frac{n\pi x}{l}\left(a_1 \cos\frac{cn\pi}{l} + a_2 \sin\frac{cn\pi}{l}t\right)$$

By first initial condition $y(x,0) = 0$

Hence $$0 = \sin\frac{n\pi x}{l}(a_1) \text{ for all x or } a_1 = 0.$$

Thus $$y(x,t) = a_2 \sin\frac{n\pi x}{l}\sin\frac{cn\pi}{l}t$$

Since this is a solution for n = 1,2,3……, therefore the sum of these solutions is also a solution.

Hence $$y(x,t) = \sum A_n \sin\frac{n\pi x}{l}\sin\frac{cn\pi}{l}t \text{ is also a solution.}$$

Now $$\frac{\partial y}{\partial t} = \sum A_n \sin\frac{n\pi x}{l}\frac{cn\pi}{l}\cos\frac{cn\pi t}{l}$$

Hence $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum A_n \frac{cn\pi}{l}\sin\frac{n\pi x}{l}$$

But $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\frac{\pi x}{l} \text{(given)}$$

Hence $$b\sin^3\frac{\pi x}{l} = \sum A_n \frac{cn\pi}{l}\sin\frac{n\pi x}{l}$$

Now $$\sin^3\frac{\pi x}{l} = \frac{1}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) \qquad [\text{because } \sin 3\theta = 3\sin\theta - 4\sin^3\theta]$$

Hence $$\frac{3b}{4}\sin\frac{\pi x}{l} - \frac{b}{4}\sin\frac{3\pi x}{l} = \sum A_n \frac{cn\pi}{l}\sin\frac{n\pi x}{l}$$

$$= A_1\frac{c\pi}{l}\sin\frac{\pi x}{l} + A_2\frac{2c\pi}{l}\sin\frac{2\pi x}{l} + A_3\frac{3c\pi}{l}\sin\frac{3\pi x}{l} + ...$$

Equating coefficients

$$A_1\frac{c\pi}{l} = \frac{3b}{4}, A_2 = 0, A_3\frac{3c\pi}{l} = -\frac{b}{4}, A_4 = A_5 = ... = 0$$

Hence $$A_1 = \frac{3lb}{4c\pi}, A_3 = -\frac{lb}{12c\pi}$$

Substituting in (1), the required solution is

$$y = \frac{3lb}{4c\pi}.\sin\frac{\pi x}{l}\sin\frac{c\pi t}{l} - \frac{lb}{12c\pi}\sin\frac{3\pi x}{l}\sin\frac{3c\pi}{l}t = \frac{lb}{12c\pi}\left[9\sin\frac{\pi x}{l}\sin\frac{c\pi t}{l} - \sin\frac{3\pi x}{l}\sin\frac{3c\pi t}{l}\right]$$

Example

Obtain the solution of heat flow equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial t^2}$$

by the method of separation of variables.

Solution

Let u(x,t) = X(x)T(t) be the solution of the equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial t^2}$$

Substituting u = XT in the equation we get

$$XT' = c^2X''T$$

or $$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = h(\text{say})\text{, where h is a constant.}$$

Hence $X'' - hX = 0$ (2)

and $T' - c^2hT = 0$ (3)

Three cases arise according as h is positive, zero or negative.

Case I **h > 0**

Let $h = k^2$

Then (2) and (3) become

$$X'' - k^2X = 0 \text{ and } T' - c^2k^2T = 0$$

Solving these differential equations, we get

$$X = c_1e^{kx} + c_2e^{-kx} \text{ and } T = c_3e^{c^2k^2t} \quad (4)$$

Case II **h = 0**

(2) and (3) become

$$X'' = 0 \text{ and } T' = 0$$

Solving $X = c_4x + c_5$ and $T = c_6$

Case III **h < 0**

Let $h = -k^2$

The (2) and (3) reduce to

$$X'' + k^2X = 0 \text{ and } T' + c^2k^2T = 0$$

Solving these differential equations

$$X = (c_8 \cos kx + c_9 \sin kx) \text{ and } T = c_{10}e^{-c^2k^2t} \quad (6)$$

Thus depending upon the values of h, we have from (4), (5) and (6), the solution of (1) in the form

$$u(x,t) = (b_1e^{kx} + b_2e^{-kx})e^{c^2k^2t} \quad (7)$$

$$u(x,t) = b_3x + b_4 \quad (8)$$

and $$u(x,t) = (b_5 \cos kx + b_6 \sin kx)e^{-c^2k^2t} \quad (9)$$

Since u(x,t) decreases as time increases, we find that solution (9) only is consistent with the physical nature of the problem. Hence (9) is the required solution.

Example

A rod of length l with insulated side is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0^0C and are kept at that temperature. Find the temperature function u(x, t).

Solution

let OA be the rod with length l. We take O as the origin and OA as x axis. The temperature u(x, t) at any point satisfies the differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial t^2}$$

By previous example, solution of this equation is

$$u(x,t) = (A\cos kx + B\sin kx)e^{-c^2k^2t} \quad (1)$$

Since the ends are cooled and kept at that temperature throughout, the boundary conditions are

$$u(0,t) = 0, u(l,t) = 0 \quad (2)$$

The rod is initially at temperature u_0; therefore the initial condition is

$$u(x, 0) = u_0. \quad (3)$$

Now, we have to find the constants in (1) subject to the boundary conditions (2) and initial condition (3).

Since u(0, t) = 0 (first boundary condition), we have from (1)

$$0 = Ae^{-c^2k^2t} \text{ or } A = 0$$

Thus (1) becomes

$$u(x,t) = B\sin kxe^{-c^2k^2t} \tag{4}$$

Also u(l, t) = 0 (second boundary condition); therefore, from (4)

$$0 = B\sin kle^{-c^2k^2t}$$

$$\therefore \quad kl = n\pi \text{ or } k = \frac{n\pi}{l} \text{ (n being an integer)}$$

The solution (4) now reduces to

$$u(x,t) = B\sin\frac{n\pi x}{l}e^{-c^2\pi^2t/l^2}$$

Since this is a solution for n = 1, 2, 3, ... the sum of these is also a solution (superposition principle).

Hence $$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}e^{-c^2n^2t/l^2}$$

is the general solution.

Now by (3) $$u(x,0) = u_0 \tag{5}$$

$$\therefore \quad u_0 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{6}$$

Since this is half range sine series for u_0

$$b_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}dx = 0 \text{ (n even)}$$

$$= \frac{4u_0}{n\pi} \text{ (n odd)}$$

Hence $$u(x,t) = \frac{4u_0}{\pi}\sum_{n=1,3,5\ldots}\frac{1}{n}\sin\frac{n\pi x}{l}\exp\left[\frac{-c^2n^2\pi^2t}{l^2}\right]$$

ASSIGNMENT

Q.1. (PTU 2018, 2 marks): Under what condition or conditions the general linear partial differential equation of second order is elliptic

Q.2. (GTU 2019, 3 marks): Form a partial differential equation from $f(xy + z^2, x + y + z) = 0$.

Answer: $(x - 2z)p - (y - 2z)q = y - x$

Q.3. (KU 2019, 7.5 marks): Form the partial differential equation of the following function

$x^2 + y^2 + (z - c)^2 = a^2$

Q.4. (KU 2019, 7.5 marks): Form the PDE from

$$x\left(z^2 - y^2\right)\frac{\partial z}{\partial x} + y\left(x^2 - z^2\right)\frac{\partial z}{\partial y} = z\left(y^2 - x^2\right)$$

Q.5. (GTU 2021, 3 marks): Form a partial differential equation for the equation $z = (x - 3)^2 + (y - 4)^2$.

Q.6. (GTU 2013, 2014, 2022, 3 marks): Form a differential equation for the equation $z = (x - 2)^2 + (y - 3)^2$.

Answer: $p^2 + q^2 = 4z$

Q.7. (PTU 2019, 2 marks): Form a partial differential equation from $z = f(x + ay) + g(x - ay)$.

Answer: $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$

Q.8. (PTU 2018, 2 marks): Form a partial differential equation from $z = f(x + y - z, xyz)$.

Q.9. (GTU 2019, 3 marks): Solve $z = px + qy + p^2q^2$

Answer: $z = ax + by + a^2b^2$

Q.10. (GTU 2020, 3 marks): Solve $pq = 1$

Answer: $z = ax + (1/a)y + c$ where a and b are arbitrary constants.

Q.11. (GTU 2020, 2021, 2022, 4 marks): Solve $p^2 - x = q^2 - y$.

Answer: $z = \dfrac{2}{3}(a + x)^{3/2} + \dfrac{2}{3}(a + y)^{3/2} + b$

Q.12. (GTU 2020, 7 marks): Solve $(p^2 + q^2)x = pz$.

Answer: $z^2 = a^2x^2 + (ay + b)^2$. Use Charpit's method.

Q.13. (GTU 2020, 4 marks): Solve $r - 2s + t = \sin(2x + 3y)$.

Answer: $z = \phi_1(y + x) + x\phi_2(y + x) - \sin(2x - 3y)$

Q.14. (GTU 2021, 3 marks): Solve $z = px + qy - 2\sqrt{(pq)}$.

Answer: $z = ax + by - 2\sqrt{(ab)}$

Q.15. (GTU 2021, 4 marks): Solve $xp + yq = 3z$.

Answer: $\phi\left(\dfrac{x}{y}, \dfrac{x^3}{z}\right) = 0$

Q.16. (GTU 2022, 4 marks): Solve $(mz - ny)p + (nx - lz)q = ly - mx$.

Answer: $x^2 + y^2 + z^2 = f(lx + my + nz)$

Q.17. (PTU 2019, 2 marks): Find solution of homogeneous partial equation r - s - 6t = 0.

Answer: $z = \phi(y - 2x) + \psi(y + 3x)$

Q.18. (GTU 2019, 7 marks): By the method of separation of variables, solve $\frac{\partial u}{\partial x} + u = \frac{\partial u}{\partial t}$ where $u(x, 0) = 4e^{-3x}$.

Answer: $u = 4e^{-3x-2t}$

Q.19. (GTU 2021, 7 marks): Solve $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, given that $u(0, y) = 8e^{-3y}$ by method of separation of variables.

Answer: $u = 8e^{-3(4x + y)}$

Q.20. (GTU 2022, 7 marks): Solve the equation $u_x = 2u_t + u$, given $u(x, 0) = 4e^{-4x}$, by method of separation of variables.

Q.21. (PTU 2018, 5 marks): Solve the partial differential equation $D^3 - 4D^2D' + 4DD'^2)z = 6\sin(3x + 2y)$.

Answer: $z = f_1(y) + f_2(y + 2x) + xf_3(y + 2x) + 2\cos(3x + 2y)$

Q.22. (PTU 2019, 5 marks): Solve the equation by method of separation of variables

$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$; given $u = 4e^{-x}$ when y = 0

Answer: $u = 4e^{(3y - 2x)/2}$

Q.23. (KU 2019, 7.5 marks): Solve the equation $2\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 5\sin(2x + y)$

Answer: $z = \phi_1\left(y + \frac{1}{2}x\right) + \phi(y + 2x) - \frac{5x}{3}\cos(2x + y)$

Q.24. (MDU 2010, GTU 2022, 4 marks): Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ given that when x = 0, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$.

Answer: $z = e^y\cos x + \sin x$

Q.25. (GTU 2021, 7 marks): Solve z = pq by Charpit's method.

Answer: $2z^{1/2} = \sqrt{ax} + \frac{y}{\sqrt{a}} + b$

Q.26. (GTU 2020, 7 marks): Solve by Charpit's method $p = (z + qy)^2$.

Answer: $yz + ax + 2\sqrt{ay} + b$

Q.27. (KU 2019, 7.5 marks): Using Charpit's method, solve (p + q)(px + qy) - 1 = 0

Answer: $z\sqrt{(1+a)} = 2\sqrt{(ax+y)} + b$