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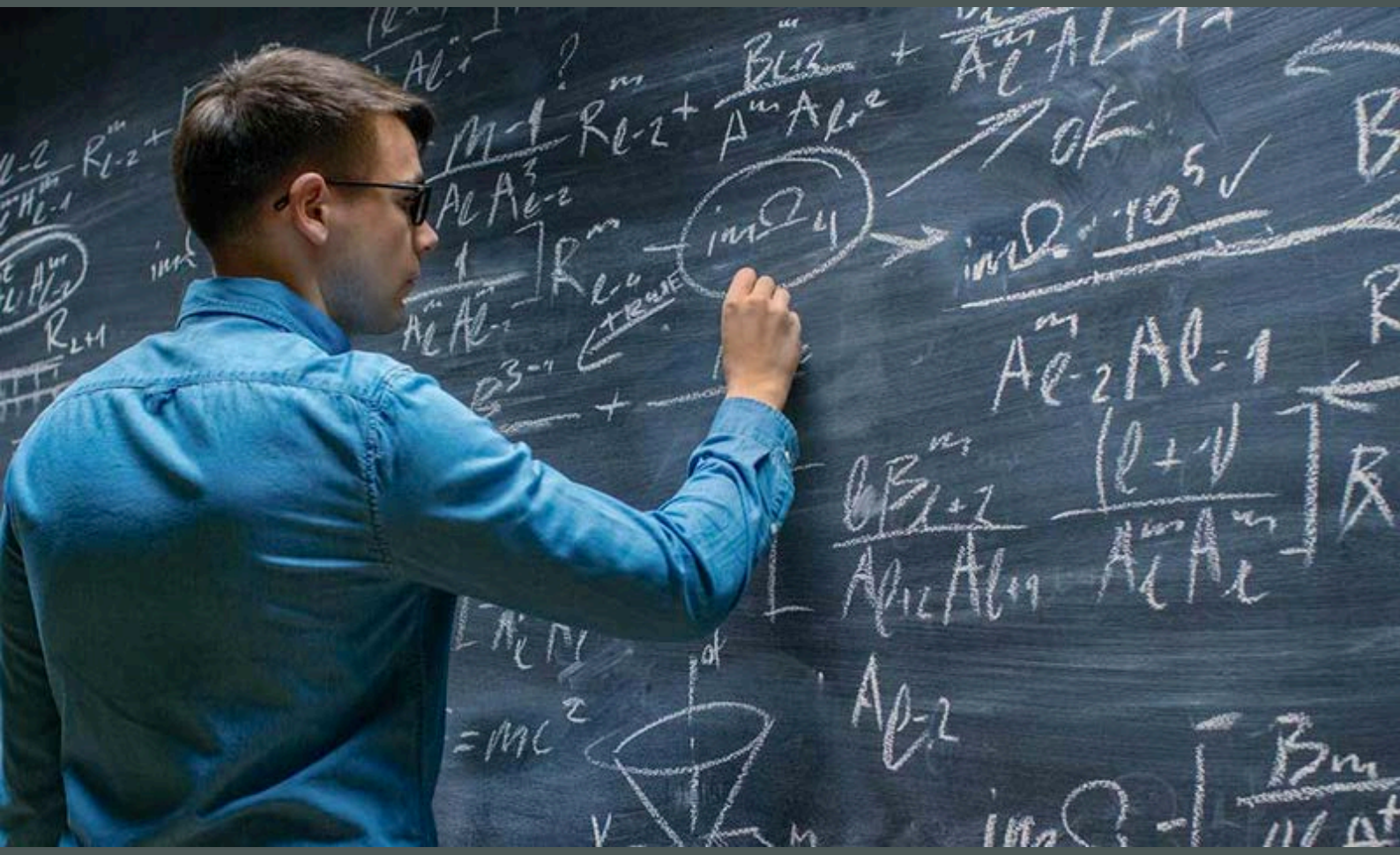

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Z-TRANSFORM

THE INVERSE Z-TRANSFORM

Mathematically, the inverse z-transform is expressed as

$$x[n] = Z^{-1}[X(z)]$$

To perform the inverse z-transform of the given expression, basically there are three methods as under:

- Long division method
- Partial fraction expansion method
- Residue method

PARTIAL FRACTION METHOD

Inversion of any rational function

$$X(z) = \frac{b_m z^m + b_{m-1} z^{m-1} + ...b_0}{z^n + a_{n-1} z^{n-1} + ...a_0}; m \le n \qquad (1)$$

can be obtained by partial fractioning it. The denominator can be written in factored form as follows.

$$D(z) = (z - p_1)(z - p_2).....(z - p_n)$$

where 1,2 are suffixes of poles of X(z).

We can then write X(z) of equation 20 in factored form as

$$X(z) = \frac{A}{z - p_1} + \frac{B}{z - p_2} + ... \qquad (2)$$

In case D(z) has m repeated roots(poles), then these poles contribute m factors as

$$\frac{A_1}{(z - p_1)} + \frac{A_2}{(z - p_1)^2} + ...\frac{A_m}{(z - p_1)^m} \qquad (3)$$

The inverse z-transform of X(z) is then obtained by looking at table 1 for the inverse transform for each factor. It is seen from following table that z-transform of every sequence has a factor z in the numerator. Therefore, the partial fraction method is applied on X(z)/z. Following examples illustrate the process.

x[n]	X(z)
$\delta(n)$	1
$\delta(n-k)$	z^{-k}
1 or u[n]	$\frac{1}{1-z^{-1}} or \frac{z}{z-1}$
a^n or $a^n u[n]$	$\frac{1}{1-az^{-1}} or \frac{z}{z-a}$
na^n or $na^n u[n]$	$\frac{az}{(z-a)^2}$
n^2a^n or $n^2a^n u[n]$	$\frac{az(z+a)}{(z-a)^3}$
$(n+1)a^n$ or $(n+1)a^n u[n]$	$\frac{z^2}{(z-a)^2}$
-u[n-1]	$\frac{z}{z-1}$
$\frac{(n+1)(n+2)a^n}{2!}$	$\frac{z^3}{(z-a)^3}$
$\frac{a^n}{n!}$ or $\frac{a^n}{n!}u[n]$	exp(a/z)
$\sin\alpha n$	$\frac{z\sin\alpha}{z^2-2z\cos\alpha+1}$
$b^n\sin\alpha n$	$\frac{b(\sin\alpha)z}{z^2-2bz\cos\alpha+b^2}$
$\cos\alpha n$	$\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$
$b^n\cos\alpha n$	$\frac{z(z-\cos\alpha)}{z^2-2bz\cos\alpha+b^2}$

Example

Find the inverse z-transform of X(z) for the following.

(a) $\dfrac{3z-5}{(z-1)(z-3)}$

(b) $\dfrac{z^2-5}{(z-1)(z-2)^2}$

Solution

(a) $X(z)=\dfrac{3z-5}{(z-1)(z-3)}$

or $\dfrac{X(z)}{z}=\dfrac{3z-5}{z(z-1)(z-3)}=\dfrac{A}{z}+\dfrac{B}{(z-1)}+\dfrac{C}{(z-3)}=\dfrac{z^2(A+B+C)+z(-4A-3B-C)+3A}{z(z-1)(z-3)}$

Comparing coefficients of numerators

$A + B + C = 0$

$-4A - 3B - C = 3$

$3A = -5$

$\therefore$ $A = -5/3, B = 1, C = 2/3$

$\therefore$ $\dfrac{X(z)}{z}=\dfrac{(-5/3)}{z}+\dfrac{1}{z-1}+\dfrac{(2/3)}{z-3}$

$\therefore$ $X(z)=\dfrac{-5}{3}+\dfrac{z}{z-1}+\dfrac{(2/3)z}{(z-3)}$

Taking inverse transform from table, we get

$$x[n]=\frac{-5}{3}\delta[n]+u[n]+\frac{2}{3}(3)^n$$

(b) $X(z)=\dfrac{z^2-5}{(z-1)(z-2)^2}$

$$\frac{X(z)}{z}=\frac{z^2-5}{z(z-1)(z-2)^2}$$

$$=\frac{A}{z}+\frac{B}{z-1}+\frac{C}{z-2}+\frac{D}{(z-2)^2}$$

$$=\frac{z^3(A+B+C)+z^2(-5A-4B-3C+D)+z(8A+4B=2C-D)+(-4A)}{z(z-1)(z-2)^2}$$

Comparing coefficients of numerators

$A + B + C = 0$

$-5A - 4B - 3C + D = 1$

$8A + 4B + 2C - D = 0$

$-4A = -5$

Solving

$A = 5/4, B = -4, C = 11/4, D = -1/2$

$$\frac{X(z)}{z} = \frac{5/4}{z} - \frac{4}{z-1} + \frac{11/4}{(z-2)} - \frac{1/2}{(z-2)^2}$$

$$X(z) = \frac{5}{4} - \frac{4z}{z-1} + \frac{(11/4)z}{z-2} - \frac{(1/2)z}{(z-2)^2}$$

Taking z-transform, we get

$$x[n] = \frac{5}{4}\delta[n] - 4u[n] + \frac{11}{4}2^n u[n] - \frac{1}{2}n2^{(n-1)}$$

Example

Find the inverse z-transform of the following in closed form

(a) $Y(z) = \dfrac{5}{1-0.2z^{-1}}$

(b) $Y(z) = \dfrac{z+z^{-1}}{z\left(1+\frac{1}{8}z^{-1}\right)}$

(c) $Y(z) = \dfrac{1}{(1-0.75z^{-1}+0.125z^{-2})}$

(d) $Y(z) = \dfrac{z^{-1}}{(1-z^{-1})(1-0.2z^{-1})^2}$

Solution

(a) $$Y(z) = \frac{5}{1-0.2z^{-1}} = \frac{5z}{z-0.2}$$

$$y[n] = Z^{-1}\left(\frac{5z}{z-0.2}\right)$$

Using z-transform from table

$$y[n] = 5(0.2)^n; n \geq 0$$

(b) Given $\dfrac{z+z^{-1}}{z\left(1+\dfrac{1}{8}z^{-1}\right)}$

or $$Y(z)=\frac{z^2+1}{z\left(z+\frac{1}{8}\right)}=\frac{z}{\left(z+\frac{1}{8}\right)}+\frac{1}{z\left(z+\frac{1}{8}\right)}$$

We obtained the inverse of the two parts of Y(z) using the z-transform table.

$$Z^{-1}\left(\frac{z}{z+\frac{1}{8}}\right)=\left(\frac{-1}{8}\right)^n;n\geq 0$$

and $$Z^{-1}\left(\frac{z^{-1}}{z+\frac{1}{8}}\right)=Z^{-1}\left\{z^{-2}\left(\frac{z}{z+\frac{1}{8}}\right)\right\}=\left(-\frac{1}{8}\right)^{n-2}$$

Then $$y[n]=\left(-\frac{1}{8}\right)^n+\left(-\frac{1}{8}\right)^{n-1}=65\left(-\frac{1}{8}\right)^n;n\geq 0$$

(c) $$Y(z)=\frac{z^2}{z^2+0.75z+0.125}$$

$$\frac{Y(z)}{z}=\frac{z}{z^2+0.75z+0.125}=\frac{z}{(z+0.25)(z+0.5)}$$

After partial fractioning, we get

$$\frac{Y(z)}{z}=\frac{2}{z+0.5}-\frac{1}{z+0.25}$$

$$Y(z)=\frac{2z}{(z+0.5)}+\frac{z}{(z+0.25)}$$

$$y[n]=2(-0.5)^n-(-0.25)^n;n\geq 0$$

(d) $$Y(z)=\frac{z^{-1}}{(1-z^{-1})(1-0.2z^{-1})^2}=\frac{z^2}{(z-1)(z-0.2)^2}$$

$$\frac{Y(z)}{z}=\frac{z}{(z-1)(z-0.2)^2}=\frac{1.5625}{(z-1)}-\frac{1.5625}{(z-0.2)}-\frac{0.25}{(z-0.2)^2}$$

Then $$Y(z)=\frac{1.5625z}{(z-1)}-\frac{1.5625z}{(z-0.2)}-\frac{0.25z}{(z-0.2)^2}$$

After taking Laplace inverse

$$y[n]=1.5625(1)^n-1.5625(0.2)^n-0.25n(0.2)^{n-1};n\geq 0$$

Problem

Use partial fraction method to find the inverse z transform of the following transfer function

$$H(z)=\frac{-4+8z^{-1}}{1+6z^{-1}+8z^{-2}}$$

Answer: $h[n]=\{-12(-4)^n+8(-2)^n\}u[n];n\geq 0$

Problem

Obtain the inverse z-transform of

$$\frac{1}{1-4z^{-1}}.$$

Answer: $(4)^n.u[n]$.

RESIDUE METHOD

The process of inverse Z-transform is the reverse process of Z-transform. The inverse Z-transform is given by a complex integral.

$$f[n]=Z^{-1}\{F(z)\}=\frac{1}{2\pi i}\oint_C F(z).z^{n-1}dz$$

where C is a simple closed curve enclosing the origin and lying outside the circle |z| = r.

Example

Find the inverse Z-transform of the function $F(z)=\frac{z}{(z-1)(z-2)}$ *using Residue theorem.*

Solution

The given function has poles at z = 1 and z = 2. The inverse Z-transform of F(z)

$$f(n)=\frac{1}{2\pi i}\oint_C \frac{z}{(z-1)(z-2)}z^{n-1}dz$$

$$f(n)=\frac{1}{2\pi i}.2\pi i(\text{sum of residues})=\sum\left[(z-z_i)F(z)z^{n-1}\right]_{z=z_i}$$

$$= \sum \left[(z - z_i).\frac{z}{(z-1)(z-2)}.z^{n-1} \right]_{z=z_i}$$

$$= \left[(z - z_i).\frac{z}{(z-1)(z-2)} z^{n-1} \right] + \left[\frac{(z-z_2)z}{(z-1)(z-2)} z^{n-1} \right]_{z=z_2}$$

$$= \left[\frac{z}{z-2} z^{n-1} \right]_{z=1} + \left[\frac{z}{(z-1)} z^{n-1} \right]_{z=2} \qquad \text{[because } z_1 = 1, z_2 = 2\text{]}$$

$$= -(1)^{n-1} + 2(2)^{n-1} = (2^n - 1)$$

DIFFERENCE EQUATIONS

Consider a relation is of the form

$$y_{n+k} + a_1 y_{n+k-1} + a_2 y_{n+k-2} + ... + a_k y_n = f(n)$$

where $a_1, a_2, \ldots, a_k$ are all constants, is called a linear difference equation with constant coefficient of order k.

Following **steps** can be use to solve:

1. Take the Z-transform of both sides of the given difference equation.
2. Using the given conditions (Initial or boundary), transpose all the terms without Y(z) to the
1. right, where $Y(z) = Z\{y_n\}$. Here $y_n = y(_n)$.
2. Simplify and find Y(z).
3. Take the inverse Z-transform of Y(z) and compute y_n, which is the required solution to the given difference equation.

☞ Remember the following:

- $Z\{y_n\} = Y(z)$
- $Z\{y_{n+k}\} = z^k \left[Y(z) - y_0 - \frac{y_1}{z} - \frac{y_2}{z} - .... - \frac{y_{k-1}}{z^{k-1}} \right]$
- $Z\{y_{n+1}\} = z\{Y(z) - y_0\}$
- $Z\{y_{n+2}\} = z^2 \left\{ Y(z) - y_0 - \frac{y_1}{z} \right\}$
- $Z\{y_{n+3}\} = z^3 \left\{ Y(z) - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2} \right\}$
- $Z\{y_{n-k}\} = z^{-k} Y(z)$

Example (AKTU 2021, 7 marks)

Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$*; given* $y_0 = 0 = y_1$.

Solution

Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

Taking Z-transform of both sides

$$Z\{y_{n+2} + 6y_{n+1} + 9y_n\} = Z\{2^n\}$$

or $$Z\{y_{n+2}\} + 6Z\{y_{n+1}\} + 9Z\{y_n\} = Z\{2^n\}$$

or $$Z^2\left[Y(z) - y_0 - \frac{y_1}{z}\right] + 6Z\left[Y(z) - y_0\right] + 9Y(z) = \frac{z}{z-2}$$

Since $y_0 = 0$ and $y_1 = 0$, we have

$$Z^2\left\{Y(z) + 6Y(z) + 9Y(z)\right\} = \frac{z}{z-2}$$

or $$(z^2 + 6z + 9)Y(z) = \frac{z}{z-2}$$

or $$Y(z) = \frac{z}{(z-2)(z^2+6z+9)} = \frac{z}{(z-2)(z+3)^2}$$

or $$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

This gives A = 1/25, B = -1/25 and C = -1/5

$\therefore$ $$\frac{Y(z)}{z} = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

Taking inverse Z-transform on both sides, we get

$$y_n = Z^{-1}\left\{Y(z)\right\} = \frac{1}{25}Z^{-1}\left(\frac{z}{z-2}\right) - \frac{1}{25}Z^{-1}\left(\frac{z}{z+3}\right) - \frac{1}{5}Z^{-1}\left\{\frac{z}{(z+3)^2}\right\}$$

$$= \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^{n-1}$$

$$= y_n = \frac{1}{25}\left[(2)^n - (-3)^n + \frac{5}{3}n(-3)^n\right]$$

Problem

Solve $y_{n+2} - 3y_{n+1} + 2y_n = 0$*; given* $y_0 = 0$ *and* $y_1 = 1$.

Answer: $y_n = (2^n - 1)$

Example

Solve $y_{n+2} - 2y_{n+1} + y_n = 3n + 5$

Solution

Taking Z transform on both sides

$$Z\{y_{n+2}\} - 2Z\{y_{n+1}\} + Z\{y_n\} = Z\{3n + 5\}$$

$$z^2\left\{Y(z) - y_0 - \frac{y_1}{z}\right\} - 2z\{Y(z) - y_0\} + Y(z) = 3\frac{z}{(z-1)^2} + 5\frac{z}{z-1}$$

or $$(z^2 - 2z + 1)Y(z) = 3\frac{z}{(z-1)^2} + 5\frac{z}{z-1}$$

or $$Y(z) = \frac{3z}{(z-1)(z^2-2z+1)} + \frac{5z}{(z-1)(z^2-2z+1)} = \frac{3z}{(z-1)^4} + \frac{5z}{(z-1)^3}$$

Taking inverse Z-transform on both sides

$$y_n = Z^{-1}\{Y(z)\} = 3Z^{-1}\left\{\frac{z}{(z-1)^4}\right\} + 5Z^{-1}\left\{\frac{z}{(z-1)^3}\right\}$$

$$= 3\binom{n}{3}u(n) + 5\binom{n}{2}u(n) \qquad \left[\because Z^{-1}\left\{\frac{a^k z}{(z-a)^{K+1}} = \binom{n}{K}a^n u[n]\right\}\right]$$

$$= \left[\frac{1}{2}n(n-1)(n-2) + \frac{5}{2}n(n-1)\right]u[n]$$

$$= \frac{1}{2}n(n-1)[n-2+5]u[n] = \frac{1}{2}n(n-1)(n+3)u[n]$$

$$= \frac{1}{2}n(n-1)(n+3) \qquad [\text{ As u[n] = 1}]$$

ASSIGNMENT

Q.1. (AKTU 2018, 2 marks): Find $Z^{-1}\left(\frac{5}{5z-1}\right)$

Q.2. (AKTU 2018, 2020, 7 marks): Find the inverse Z-transform of (F(z) where F(z) is given by

(i) $\frac{z}{(z+2)(z+3)}$

(ii) $\frac{7z-11z^2}{(z-1)(z-2)(z+3)}$

Q.3. (AKTU 2019, 3.5 marks): By residue method, find the inverse Z-transform of $\frac{z}{z^2+7z+10}$.

Answer: $f(n)=\frac{1}{3}(-2)^n-\frac{1}{3}(-5)^n$

Q.4. (AKTU 2021, 7 marks): Solve by Z-transform, the differential equality $y_{k+2} + 6y_{k+1} + 9y_k = 2^k$; ($y_0 = y_1 = 0$).

Answer: Solved in this module.