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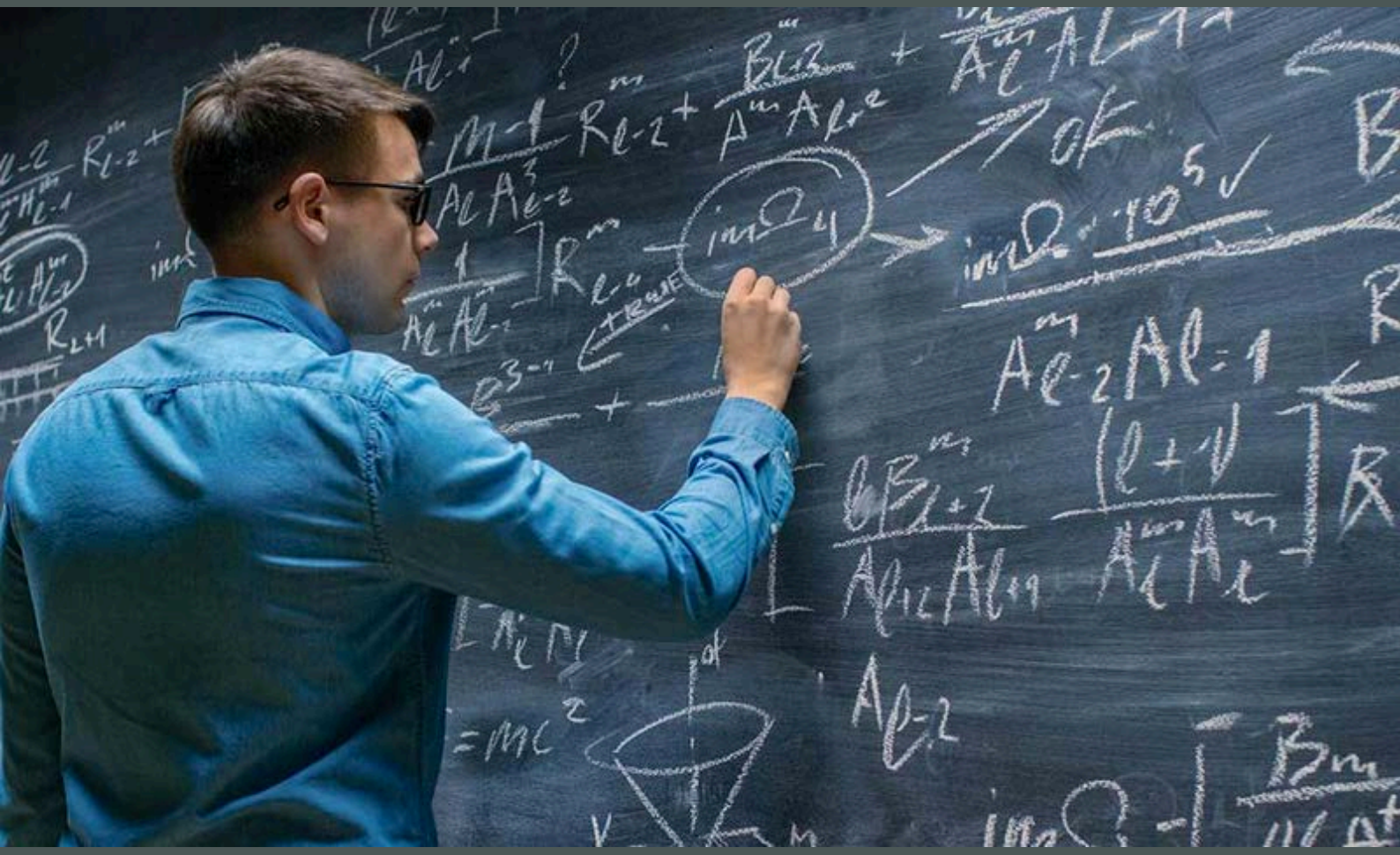

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LAPLACE TRANSFORM

Let f(t) be the function defined for all positive values of t, then

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

provided the integral exists, is called Laplace Transform of f(t).

It is denoted as

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt$$

TABLE OF LAPLACE TRANSFORM

Some of the transform pairs have already been obtained in the examples earlier. In table some commonly used transform pairs are presented. You should memorize this table.

Basic Laplace Transform

f(t)	**F(s)**
L(1)	1/s
t	$1/s^2$
$L(t^n)$	$\frac{n!}{s^n+1}$
$L(e^{-at})$	1/(s + a)
$L(te^{-at})$	$1/(s + a)^2$
$L(t^ne^{-at})$	$\frac{n!}{(s+a)^{n+1}}$
L(cos at)	$\frac{s}{s^2+a^2}$
L(sin at)	$\frac{a}{s^2+a^2}$
$L(e^{at}\cos bt)$	$\frac{s-a}{(s-a)^2+b^2}$

$L(e^{at}\sin bt)$	$\dfrac{b}{(s-a)^2+b^2}$

PROPERTIES

Multiplication by t^n

$$L\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

If n = 1, $$L\left\{tf(t)\right\} = -\frac{d}{ds} F(s)$$

Linearity property

$$L\left[af_1(t)+bf_2(t)\right] = aL\left[f_1(t)\right]+b\left[f_2(t)\right]$$

Change of scale property

$$L\left\{f(at)\right\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

First shifting theorem

$$L\left[e^{at} f(t)\right] = F(s-a)$$

Second shifting theorem

$$L[f(t-a).u(t-a)] = e^{-as}F(s)$$

IMPORTANT TRIGONOMETRICAL FORMULA

- $\sin x \cdot \cos y = [\sin(x + y) + \sin(x - y)]/2$
- $\cos x \cdot \cos y = [\cos(x + y) + \cos(x - y)]/2$
- $\sin x \cdot \sin y = [\cos(x - y) - \cos(x + y)]/2$
- $\sin x + \sin y = 2[\sin((x + y)/2)\cos((x - y)/2)]$
- $\sin x - \sin y = 2[\cos((x + y)/2)\sin((x - y)/2)]$
- $\cos x + \cos y = 2[\cos((x + y)/2)\cos((x - y)/2)]$
- $\cos x - \cos y = -2[\sin((x + y)/2)\sin((x - y)/2)]$
- $(\sin x + \cos x)^2 = \sin 2x + 1$
- $(\sin x - \cos x)^2 = 1 - \sin 2x$



- $\sin 2x = 2\sin(x) \bullet \cos(x)$
- $\cos 2x = 1 - 2\sin 2x$
- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
- $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
- $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
- $\cos^2\left(\frac{x}{2}\right) = \frac{1}{2}(1+\cos x)$
- $\sin^2\left(\frac{x}{2}\right) = \frac{1}{2}(1-\cos x)$
- $\sin 3x = 3\sin x - 4\sin^3 x$
- $\cos 3x = 4\cos^3 x - 3\cos x$
- $\sinh x = \frac{e^x - e^{-x}}{2}$
- $\cosh x = \frac{e^x + e^{-x}}{2}$
- $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

STANDARD INTEGRAL FORMULA

Formula	Example
$\int x^n dx = \frac{x^{n+1}}{n+1} \; (n \neq 1)$	$\int x^{3/2} dx = \frac{x^{(3/2)+1}}{(3/2)+1} = \frac{2}{5}x^{5/2}$
$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a}$ where $n \neq -1$	$\int (2x+3)^5 dx = \frac{(2x+3)^6}{2x6} = \frac{(2x+3)^6}{12}$
$\int \frac{1}{x} dx = \log x$	
$\int \frac{dx}{(ax+b)} = \frac{1}{a}\log(ax+b)$	$\int \frac{1}{2x+7} dx = \frac{1}{2}\log(2x+7)$
$\int e^x dx = e^x$	
$\int e^{ax} dx = \frac{1}{a}e^{ax}$	

$\int a^x dx = \frac{a^x}{\log a}$	$\int 5^x dx = \frac{5^x}{\log 5}$
$\int a^{kx} dx = \frac{a^{kx}}{k \log a}$	
$\int \sin x.dx = -\cos x$	
$\int \cos x.dx = \sin x$	
$\int \tan x.dx = \log \sec x = -\log \cos x$	
$\int \cot x.dx = \log \sin x$	
$\int \sec x.dx = \log(\sec x + \tan x) = \log\left(\frac{\pi}{2} + \frac{x}{2}\right)$	
$\int \cos ecx.dx = \log(\cos ecx - \cot x) = \log . \tan \frac{x}{2}$	
$\int \sec^2 x.dx = \tan x$	
$\int \cos ec^2 x.dx = -\cot x$	
$\int \sec x. \tan x.dx = \sec x$	
$\int \cos ecx. \cot x.dx = -\cos ecx$	
$\int \cosh x.dx = \sinh x$	
$\int \sinh dx = \cosh x$	
$\int af(x)dx = a\int f(x)dx$	$\int \sin 3xdx = -\frac{1}{3}\cos 3x$

Example

Find the Laplace transform of sin2t.sin3t.

Solution

Simplification



$$\sin a.\sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

Apply Laplace formula

$$L\left(\sin 2t.\sin 3t\right) = \frac{1}{2}L\left[\cos t - \cos 5t\right]$$

$$\frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right] = \frac{12s}{(s^2+1)(s^2+25)}$$

Problem

Find the Laplace transform of $\cos^2 2t$.

Answer: $\frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$

Hint: $\cos^2(2t) = \frac{1}{2}\left(1+\cos 4t\right)$

Example

Find Laplace transform of $\sin^3 2t$.

Solution

Simplification

We know $\sin 3x = 3\sin x - 4\sin^3 x$

$\therefore$ $\sin 6t = 3\sin 2t - 4\sin^3 2t$

$\therefore$ $\sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$

Apply Laplace formula

$$L\left(\sin^3 2t\right) = \frac{3}{4}L(\sin 2t) - \frac{1}{4}(\sin 6t)$$

$$= \frac{3}{4}.\frac{2}{s^2+2^2} - \frac{1}{4}.\frac{2}{s^2+6^2}$$

$$= \frac{48}{(s^2+4)(s^2+36)}$$



Problem

Find Laplace transform of 1 + sin2t

Answer: $\frac{1}{s}+\frac{2}{s^2+4}$

Example (PTU 2019, JNTU 2014)

Find Laplace transform of (i) f(t) = e^{-2t}tcost (ii) f(t) = t^2e^{-2t}cost.

Solution

(i) We know that $L\{\cos t\}=\frac{s}{s^2+1}$

Using the formula

$$L\{tf(t)\}=-\frac{d}{ds}F(s)$$

$$L\{t\cos t\}=-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)=\frac{s^2-1}{(s^2+1)^2}$$

Now, using first shifting property

$$L\{e^{-2t}t\cos t\}=\frac{(s+2)^2-1}{[(s+2)^2+1]^2}=\frac{s^2+4s+3}{(s^2+4s+5)^2}$$

(ii) $$L\{t\cos t\}=-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)=\frac{s^2-1}{(s^2+1)^2}$$ (just obtained)

$\therefore$ $$L\{t^2\cos t\}=-\frac{d}{ds}\left(\frac{s^2-1}{(s^2+1)^2}\right)$$

Using first shifting property

$$L\{t^2e^{-2t}\cos t\}=2\left(\frac{s^3+10s^2+25s+22}{(s^2+4s+5)^3}\right)$$

Example (JNTU 2006, 2007)

Find Laplace transform of (te^{-t}sin2t)

Solution

Using the formula

$$L\{tf(t)\} = -\frac{d}{ds}F(s)$$

We get $L(t\sin 2t) = (-1)\frac{d}{ds}F(s)$

$\therefore$ $L(t\sin 2t) = (-1)\frac{d}{ds}\left[\frac{2}{s^2+4}\right] = (-2)\left[\frac{-1}{(s^2+4)}.2s\right] = \frac{4s}{(s^2+4)^2}$

$\therefore$ Using first shifting theorem

$$L(e^{-t}t\sin 2t) = \left[\frac{4s(s+1)}{[(s+1)^2+4]^2}\right] = \frac{4(s+1)}{(s^2+2s+5)^2}$$

Example (PTU 2009)

Find Laplace transform of (i) $\int_0^t \frac{\sin t}{t}dt$ *(ii)* $\int_0^t \frac{\sin t}{t}dt$

Solution

(i) $L(\sin t) = \frac{1}{s^2+1}$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}ds = \frac{\pi}{2} - \tan^{-1}s = \cos^{-1}s$$

$\therefore$ $L\left(\int_0^t \frac{\sin t}{t}dt\right) = \frac{1}{s}\cot^{-1}s$

(ii) $L\left(\frac{\sin t}{t}\right) = \cot^{-1}s$

$\therefore$ $L\left(e^t\frac{\sin t}{t}\right) = \cot^{-1}(s-1)$

$\therefore$ $L\left(\int_0^t \frac{\sin t}{t}dt\right) = \frac{1}{s}\cot^{-1}(s-1)$

Example (PTU 2020, 5 marks)

Evaluate Laplace transform of $\int_0^\infty t^3e^{-t}\sin t dt$



Solution

Here $L(\sin t) = \frac{1}{(s^2+1)} = F(s)$

$\therefore$ $L(t^3 \sin t) = (-1)^3 \frac{d^3}{ds^3} F(s) = -\frac{d^3}{ds^3}(s^2+1)^{-1}$

$= = -\frac{d^2}{ds^2}\left[\frac{d}{ds}(s^2+1)^{-1}\right] = -\frac{d^2}{ds^2}\left[-(s^2+1)^{-2} x 2s\right]$

$= = 2\frac{d}{ds}\left[\frac{1x(s^2+1)^2 - sx2(s^2+1)x2s}{(s^2+1)^4}\right] = 2\frac{d}{ds}.\frac{1-3s^2}{(1+s^2)^3}$

$= 2\frac{(-6s)(1+s^2)^3 - 3(1+s^2)^2 x2s(1-3s^2)}{(1+s^2)^6} = 12s\frac{1+s^2+1-3s^2}{(1+s^2)^4}$

$= \frac{24s(s^2-1)}{(1+s^2)^4}$

or $\int_0^\infty e^{-st}\{t^3 \sin t\}\, dt = \frac{24s(s^2-1)}{(1+s^2)^4}$

Taking limit as s $\to$ 1 on both sides, we get

$$\int_0^\infty t^3 e^{-t} \sin t dt = 0$$

Example (JNTU, WBTU)

Find Laplace transform of

$$f(t) = \begin{cases} 1 & 0 < t \le 1 \\ t & 1 < t \le 2 \\ 0 & t > 2 \end{cases}$$

Solution

$$L[f(t)] = \int_0^1 e^{-st}.1dt + \int_1^2 e^{-st}.tdt + \int_2^\infty e^{-st}(0)dt$$

$$= \left|\frac{e^{-st}}{-s}\right|_0^1 + \left|t.\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_1^2$$



$$= \frac{1-e^{-s}}{s}\left[\left(-\frac{2e^{-2s}}{s}-\frac{e^{-2s}}{s^2}\right)-\left(\frac{e^{-s}}{-s}-\frac{e^{-s}}{s^2}\right)\right]$$

$$= \frac{1}{s}-\frac{2e^{-2s}}{s}+\frac{e^{-s}}{s^2}-\frac{e^{-2s}}{s^2}$$

Example

Find the Laplace transform of the function

$$f(t)=\begin{cases} t-1 & 1<t<2 \\ 3-t & 2<t<3 \end{cases}$$

Solution

$$L[f(t)]=\int_1^3 f(t)e^{-st}dt=\left[\int_1^2 (t-1)e^{-st}dt+\int_2^3 (3-t)e^{-st}dt\right]$$

$$= \left[\left\{(t-1)\frac{e^{-st}}{-s}-\frac{e^{-st}}{(-s)^2}\right\}_1^2\right]+\left\{(3-t)\frac{e^{-st}}{-s}+\frac{e^{-st}}{(-s)^2}\right\}_2^3$$

$$= \left[\left\{\frac{e^{-2s}}{-s}-\frac{e^{-2s}}{s^2}+\frac{e^{-s}}{s^2}\right\}+\left\{\frac{e^{-3s}}{s^2}-\frac{e^{-2s}}{-s}-\frac{e^{-2s}}{s^2}\right\}\right]$$

$$= \left[-\frac{e^{-2s}}{s}-\frac{e^{-2s}}{s^2}+\frac{e^{-s}}{s^2}+\frac{e^{-3s}}{s^2}+\frac{e^{-2s}}{s}-\frac{e^{-2s}}{s^2}\right]$$

$$= \left[\frac{1}{s^2}(-e^{-2s}+e^{-s}+e^{-3s}-e^{-2s})\right]$$

$$= \frac{1}{s^2}\left[e^{-s}-2e^{-2s}+e^{-3s}\right]$$

Problem

Find the Laplace transform of

$$f(t)=\begin{cases} 1 & 0\le t<1 \\ t & 1\le t<2 \\ t^2 & 2\le t<\infty \end{cases}$$

Answer: $\frac{1}{s}+\frac{2}{s}e^{-2s}+\frac{e^{-s}}{s^2}+\frac{3}{s^2}e^{-2s}+\frac{2}{s^3}e^{-2s}$

Example

Find $L\left\{\frac{\sin at}{t}\right\}$ given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left\{\frac{1}{s}\right\}$

Solution

Using change of scale property

$$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\tan^{-1}\left\{\frac{1}{(s/a)}\right\}$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right)$$

$$\therefore \quad L\left(\frac{\sin at}{t}\right) = \tan^{-1}\left(\frac{a}{s}\right)$$

LAPLACE TRANSFORMS OF PERIODIC FUNCTIONS

If f(t) is a periodic function with period T, i.e. f(t + T) = f(t)

Then $$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-st}}$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt \quad \text{After simplifying}$$

Example (KU 2019, 7.5 marks)

Find the Laplace transform of the periodic function

$$f(t) = \sin \omega t \quad 0 < t < \pi/\omega$$
$$0 \qquad \pi/\omega < t < 2\pi/\omega$$

Solution

Since f(t) is the periodic function with period 2π/ω.

$$\therefore \quad L(f(t)) = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin \omega t dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}.0dt\right]$$



$$= \frac{1}{1-e^{-2\pi s/\omega}} \left| \frac{e^{-st}\left(-s\sin\omega t - \omega\cos\omega t\right)}{s^2+\omega^2} \right|_0^{\pi/\omega}$$

$$= \frac{\omega e^{-\pi/\omega} + \omega}{\left(1-e^{-2\omega s/\omega}\right)\left(s^2+\omega^2\right)}$$

$$= \frac{\omega}{\left(1-e^{-\pi s/\omega}\right)\left(s^2+\omega^2\right)}$$

Example

Find the Laplace transform of the waveform

$$f(t) = \left(\frac{2t}{3}\right), 0 \le t \le 3$$

Solution

$$L\left[f(t)\right] = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)dt$$

$$L\left(\frac{2t}{3}\right) = \frac{1}{1-e^{-3s}} \int_0^3 e^{-sT} \left(\frac{2}{3}t\right)dt$$

$$= \frac{1}{1-e^{-3s}} \frac{2}{3}\left[\frac{te^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$= \frac{2}{3}.\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{2}{3}.\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s} + \frac{1-e^{-3s}}{s^2}\right]$$

$$= \frac{2e^{-3s}}{-s\left(1-e^{-3s}\right)} + \frac{2}{3s^2}$$

Problem

Find the Laplace Transform of the periodic function

$$f(t) = \frac{kt}{T} \quad 0 < t < T,\ f(t+T) = f(t)$$



Answer: $-\dfrac{ke^{-sT}}{s\left(1-e^{-st}\right)}+\dfrac{k}{Ts^2}$

Problem

Draw the graph of the following periodic function and find its laplace transform:

$$f(t)=\begin{cases}t & 0<t\le a\\ 2a-t & a<t<2a\end{cases}$$

Answer: $\dfrac{1}{s^2}\left[\dfrac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right]=\dfrac{1}{s^2}\tanh\dfrac{as}{2}$

Example

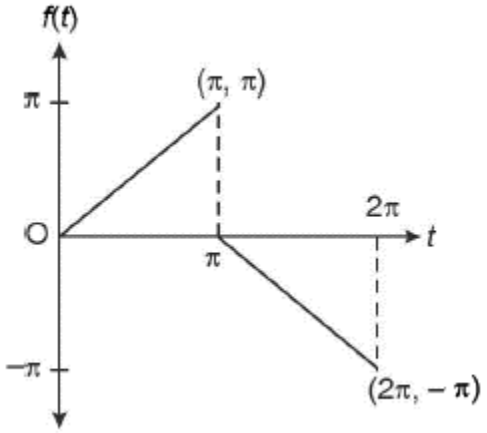
Draw the graph of the periodic function and find laplace transform of

$$f(t)=\begin{cases}t & 0<t<\pi\\ \pi-t & \pi<t<2\pi\end{cases}$$

Solution

Period = 2π

Graph is given below.



Now $$L\left[f(t)\right]=\frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi}e^{-st}tdt+\int_{\pi}^{2\pi}e^{-st}(\pi-t)dt\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\left|t\left(\frac{e^{-st}}{-s}\right)-1\left(\frac{e^{-st}}{s^2}\right)\right|_0^{\pi}+\left|(\pi-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{-\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{\pi}{s}\left(e^{-2\pi s} - e^{-\pi s}\right)\frac{1}{s^2}\left(1 + e^{-2\pi s} - 2e^{-\pi s}\right)\right]$$

LAPLACE TRANSFORM OF THE DERIVATIVES

- L[f'(t)=sL[f(t)] - f(0) where L[f(t)] = F(s)
- $L[f^n(t)] = s^nL[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) + ... - f^{n-1}(0)$

Example

Given $L\left(2\sqrt{\frac{t}{\pi}}\right) = \frac{1}{s^{3/2}}$, *show that* $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$

Solution

Let $F(t) = 2\sqrt{\frac{t}{\pi}}$

$\therefore$ $F'(t) = \frac{1}{\sqrt{\pi t}}$

and F(0) = 0

$\therefore$ $L\{F'(t)\} = sL\{F(t)\} - F(0)$

$= s.\frac{1}{s^{3/2}} - 0 = \frac{1}{\sqrt{s}}$

$\therefore$ $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$

LAPLACE TRANSFORM OF INTEGRAL OF F(T)

It is given by

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$$

where L[f(t)] = F(s)



LAPLACE TRABNSFORM OF T.F(T)

It is given by

$$L[t^n f(t)] = (-1^n)\frac{d^n}{ds^n}[F(s)]$$

DIVISION BY "T"

It is given by

$$L\left[\frac{1}{t} f(t)\right] = \int_0^\infty F(s)ds$$

Example

Find the laplace transform of (1 - e^t)/t.

Solution

$$L(1-e^t) = L(1) - L(e^t) = \frac{1}{s} - \frac{1}{s-1}$$

$$\therefore \quad L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right)ds = \left|\log s - \log(s-1)\right|_s^\infty$$

$$= = \left|\log\left(\frac{s}{s-1}\right)\right|_s^\infty = -\log\left|\frac{1}{1-(1/s)}\right| = \log\left(\frac{s-1}{s}\right)$$

Example

Find the Laplace transform of $f(t) = \int_0^t \frac{\sin at}{t}dt$

Solution

$$L(\sin at) = \frac{a}{s^2 + a^2}$$

$$L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2 + a^2} ds = \left[\tan^{-1}\frac{s}{a}\right]$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cos^{-1}\frac{s}{a}$$

$$\therefore \qquad L\left[\int_0^t \frac{\sin at}{t}\,dt\right] = \frac{1}{s}F(s) = \frac{1}{s}\cot^{-1}\frac{s}{a}$$

Example

Find the Laplace transform of $\dfrac{\cos at - \cos bt}{t}$

Solution

$$f(t) = \frac{\cos at - \cos bt}{t}$$

We know $\quad L(\cos at - \cos bt) = L(\cos at) - L(\cos bt)$

$$= \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}$$

Now $\quad L\left(\dfrac{\cos at - \cos bt}{t}\right) = \int_s^\infty \left(\dfrac{s}{s^2 + a^2} - \dfrac{s}{s^2 + b^2}\right) ds$

$$= \left[\frac{1}{2}\log\left(s^2 + a^2\right) - \frac{1}{2}\log\left(s^2 + b^2\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2 + a^2}{s^2 + b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{1 + \dfrac{a^2}{s^2}}{1 + \dfrac{b^2}{s^2}}\right]$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\frac{1 + \dfrac{a^2}{s^2}}{1 + \dfrac{b^2}{s^2}}$$

$$= 0 - \frac{1}{2\log\dfrac{s^2 + a^2}{s^2 + b^2}} \qquad [\log 1 = 0]$$

$$= \frac{1}{2}\log\frac{s^2 + b^2}{s^2 + a^2}$$



Problem

Find the Laplace transform of $\frac{\cos at - \cos bt}{t} + t \sin at$

Answer: $\log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2} + \frac{2as}{(s^2+a^2)^2}$

Problem

If $f(t) = \frac{e^{at} - \cos bt}{t}$, find the Laplace transform of f(t).

Answer: $\frac{1}{2}\left[\log\left(\frac{s^2+b^2}{(s-a)^2}\right)\right]$

Example

Evaluate $\left[e^{-t}\int_0^t \frac{\sin t}{t} dt\right]$

Solution

$$L(\sin t) = \frac{1}{s^2+1}$$

$$\therefore \quad L\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{1}{s^2+1} ds$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

Now, using shifting property

$$L\left[e^{-t}\left(\int_0^t \frac{\sin t}{t}\right) dt\right] = \frac{1}{s+1}\cot^{-1}(s+1)$$

Example

Evaluate $L\left[t\int_0^t \frac{e^{-t}\sin t}{t} dt\right]$



Solutuion

$$L\left(\frac{\sin t}{t}\right) = \cot^{-1} s$$

$\therefore$ $$L\left(e^{-t}.\frac{\sin t}{t}\right) = \cot^{-1} s$$

Now $$L\left(e^{-t}.\frac{\sin t}{t}dt\right) = \cot^{-1}(s+1)$$

$$L\left(\int_0^t e^{-t}\frac{\sin t}{t}dt\right) = \frac{1}{s}\cot^{-1}(s+1)$$

$\therefore$ $$L\left(t.\int_0^t e^{-t}\frac{\sin t}{t}dt\right) = -\frac{d}{ds}\left(\frac{\cot^{-1}(s+1)}{s}\right)$$

$$= -\frac{s\left[\frac{-1}{1+(s+1)^2}\right]-\cot^{-1}(s+1)}{s^2}$$

$$= \frac{s+(s^2+2s+2)\cot^{-1}(s+1)}{s^2(s^2+2s+2)}$$

INITIAL VALUE THEOREM

If the function f(t) and its derivative f'(t) are Laplace transformable then

$$f(0) = \lim_{s\to\infty} sF(s)$$

Proof

We know that L[f'(t)] = sF(s) – f(0) [Derivative formula]

By taking the limit s $\to \infty$ on both sides

$$\lim_{s\to\infty} L[f'(t)] = \lim_{s\to\infty}[sF(s) - f(0)]$$

or $$\lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to\infty}[sF(s) - f(0)]$$

As s$\to\infty$, the integration on L.H.S. becomes zero.

i.e. $$\int_0^\infty \lim_{s\to\infty}[f(t)e^{-st}]dt = 0$$

$$0 = \lim_{s\to\infty} sF(s) - f(0)$$



i.e. $f(0) = \lim_{s\to\infty} sF(s)$

$\therefore$ $sF(s) = \underset{t\to\infty}{Lim}\, f(t)$

This is initial value theorem.

FINAL VALUE THEOREM

If f(t) and f'(t) are Laplace transformable then

$$f(\infty) = \lim_{s\to 0} sF(s)$$

Proof

We know that L[f(t)] = sF(s) – f(0)

By taking the limit s $\to$0 on both sides, we have

$$\lim_{s\to 0} L[f(t)] = \lim_{s\to 0}[sF(s) - f(0)]$$

or $$\lim_{s\to 0}\int_0^\infty f(t)e^{-st}dt = \lim_{s\to 0}[sF(s) - f(0)]$$

$$\int_0^\infty f(t)dt = \lim_{s\to 0}[sF(s) - f(0)]$$

But $$\left(f(t)\right)_0^\infty = \lim_{t\to\infty} f(t) - \lim_{t\to 0}$$

Hence $$\lim_{t\to\infty} f(t) - \lim_{t\to 0} = \lim_{s\to 0} sF(s) - f(0)$$

Since f(0) is not a function of S, it gets cancelled from both sides

Hence $$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$\therefore$ $sF(s) = \underset{t\to\infty}{Lim}\, f(t)$

Example

Find the final value of the function whose Laplace transform is

$$F(s) = \frac{s+9}{s^2+7s+3}$$

Solution

$\therefore$ $$f(0) = \underset{s\to\infty}{Lt}\, sF(s) = \underset{s\to\infty}{Lt}\, s\left(\frac{s+9}{s^2+7s+3}\right)$$



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Dividing both numerator and denominator by s^2

$$f(0) = \underset{s\to\infty}{Lt}\left(\frac{1+\frac{9}{s}}{1+\frac{7}{s}+\frac{3}{s^2}}\right) = 1$$

and $$f(\infty) = \underset{s\to 0}{Lt}\, sF(s) = \underset{s\to 0}{Lt}\, s\left(\frac{s+9}{s^2+7s+3}\right)$$

$$= \underset{s\to 0}{Lt}\left(\frac{s+9}{s+7+\frac{3}{s}}\right) = \frac{9}{\infty} = 0$$

Problem

Find the final value of the function whose Laplace transform is

$$F(s) = \frac{s+6}{s(s+3)}$$

Answer: 2

Problem

Find the initial value of the continuous signal if its Laplace transform is given as

$$X(s) = \frac{2s+1}{s^2-1}$$

Answer: 2

INVERSE TRANSFORMS

If F(s) is the Laplace Transform of a function f(t), then f (t) is known as **Inverse Laplace Transform**.

If L[f(t)] = F(s), then $L^{-1}[F(s)] = f(t)$ where L^{-1} is called the Inverse Laplace Transform operator.

Important formulae

1. $L^{-1}\left(\frac{1}{s}\right) = 1$	2. $L^{-1}\frac{1}{s^n} = \frac{t^{n-1}}{(n-1)!}$



3. $L^{-1}\dfrac{1}{s-a}=e^{at}$	4. $L^{-1}\dfrac{s}{s^2-a^2}=\cosh at$
5. $L^{-1}\dfrac{1}{s^2-a^2}=\dfrac{1}{a}\sin at$	6. $L^{-1}\dfrac{1}{s^2+a^2}=\dfrac{1}{a}\sin at$
7. $L^{-1}\dfrac{s}{s^2+a^2}=\cos at$	8. $L^{-1}F(s-a)=e^{at}f(t)$
9. $L^{-1}\dfrac{1}{(s-a)^2+b^2}=\dfrac{1}{b}e^{at}\sin bt$	10. $L^{-1}\dfrac{s-a}{(s-a)^2+b^2}=e^{at}\cos bt$
11. $L^{-1}\dfrac{1}{(s-a)^2-b^2}=\dfrac{1}{b}e^{at}\sinh bt$	12. $L^{-1}\dfrac{s-a}{(s-a)^2-b^2}=e^{at}\cosh bt$
13. $L^{-1}\dfrac{1}{(s^2+a^2)}=\dfrac{1}{2a^3}(\sin at-at\cos at)$	14. $L^{-1}\dfrac{s}{(s^2+a^2)}=\dfrac{1}{2a}t\sin at$
15. $L^{-1}\dfrac{s^2-a^2}{(s^2+a^2)^2}=t\cos at$	16. $L^{-1}(1)=s(t)$
17. $L^{-1}\dfrac{s^2}{(s^2+a^2)^2}=\dfrac{1}{2a}[\sin at+at\cos at]$	18. $L^{-1}\left(\dfrac{1}{s}F(s)\right)=\int_0^t f(t)dt$
19. $L^{-1}(1)=\delta(t)$	

FINDING ROOTS OF A CUBIC POLYNOMIAL

Let the polynomial be $f(x) = x^3 - 5x^2 - 2x + 24 = 0$

Lets find a number which makes it zero.

If we put $x = -2$, we get $f(x) = 0$

Hence $(x + 2)$ is a factor.

Divide $x^3 - 5x^2 - 2x + 24$ by $(x + 2)$. Now we will use long division.



$$
\begin{array}{r}
x^2-7x+12 \\
x+2\overline{)\,x^3-5x^2-2x+24} \\
x^3+2x^2 \\
\underline{-7x^2-2x} \\
-7x^2-14x \\
\underline{12x+24} \\
12x+24 \\
\underline{0}
\end{array}
$$

The factors of $x^3 - 5x^2 - 2x + 24$ are $(x + 2)$ and $(x^2 - 7x + 12)$.

Again the factors of $(x^2 - 7x + 12)$ are $(x - 3)$ and $(x - 4)$ by trial and error.

The factors of quadratic equation $(x^2 - 7x + 12)$ can also be calculeted from formula:

The roots of quadratic equation $ax^2 + bx + c$ are

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Hence, in given equation $x^2 - 7x + 12$

$a = 1$; $b = -7$ and $c = 12$

Puting these values in above formula

$x = 3$ and $x = 4$

Hence, $(x - 3)$ and $(x - 4)$ are other factors.

So three factors are $(x + 2)$, $(x - 3)$ and $(x - 4)$.

Example

Find inverse Laplace transform of the following:

(i) $\dfrac{1}{s^2-9}$

(ii) $\dfrac{1}{(s-2)^2+1}$

(iii) $\dfrac{1}{(s+3)^2-4}$

(iv) $\dfrac{s+2}{(s+2)^2-25}$



Solution

(i) $L^{-1}\dfrac{1}{s-2}=e^{2t}$

(ii) $L^{-1}\dfrac{1}{(s-2)^2+1}=e^{2t}\sin t$

(iii) $L^{-1}\dfrac{1}{(s+3)^2-4}=\dfrac{1}{2}\dfrac{2}{(s+3)^2-(2)^2}=\dfrac{1}{2}e^{-3t}\sinh 2t$

(iv) $L^{-1}\dfrac{s+2}{(s+2)^2-25}=L^{-1}\dfrac{(s+2)}{(s+2)^2-(5)^2}=e^{-2t}\cosh 5t$

Example

Find inverse Laplace transform of $\dfrac{s^2-3s+4}{s^3}$

Solution

$$L^{-1}\frac{s^2-3s+4}{s^3}=L^{-1}\left[\frac{s^2}{s^3}-\frac{3s}{s^3}+\frac{4}{s^3}\right]=L^{-1}\left[\frac{1}{s}-3.\frac{1}{s^2}+4.\frac{1}{s^3}\right]$$

$$=1-3t+4\frac{t^2}{2!}$$

$$=1-3t+2t^2$$

Example

Obtain the inverse Laplace transform of

$$F(s)=\frac{s^2+3s+1}{s(s+1)(s+2)}$$

Solution

Partial fraction

Let us first do partial fractioning of F(s)

$$F(s)=\frac{s^2+3s+1}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

[This is a case of distinct real poles]



$$= \frac{A(s+1)(s+2)+B(s)(s+2)+C(s)(s+1)}{s(s+1)(s+2)}$$

$$= \frac{A(s^2+2s+2)+B(s^2+2s)+C(s^2+s)}{s(s+1)(s+2)}$$

$$= \frac{(A+B+C)s^2+(3A+2B+C)s+2A}{s(s+1)(s+2)}$$

Comparing coefficients of numerators on both sides

A + B + C = 1

3A +2B + C = 3

2A = 1

Solving A = 0.5, B = 1 and C = -0.5

$$F(s) = \frac{s^2+3s+1}{s(s+1)(s+2)} = \frac{0.5}{s} + \frac{1}{s+1} - \frac{0.5}{s+2}$$

Inverse Laplace Transform

Using table, we get

$$f(t) = 0.5 + e^{-t} - \frac{1}{2}e^{-2t}$$

Example

Obtain inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s+1)}$$

Solution

Partial fraction

Let us first do partial fractioning of F(s)

$$F(s) = \frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+1)}$$ [This is a case of repeated poles]

$$= \frac{As(s+1)+B(s+1)+C(s^2)}{s^2(s+1)}$$



$$= \frac{A(s^2+s)+B(s+1)+Cs^2}{s^2(s+1)}$$

$$= \frac{(A+C)s^2+(A+B)s+B}{s^2(s+1)}$$

Comparing coefficients of numerators on both sides

B = 1

A + B = 0

A + C = 0

Solving

B = 1

A = -1

C = 1

$\therefore$ Inverse of $F(s) = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$

Inverse Laplace transform

From table, we get

$$f(t) = (t - 1 + e^{-t})$$

Example

Obtain the inverse Laplace transform of

$$F(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s+2)}$$

Solution

Partial Fractioning

$$\frac{2s^2+6s+6}{(s+2)(s^2+2s+s)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+2} = \frac{A(s^2+2s+2)+(Bs+C)(s+2)}{(s+2)(s^2+2s+2)}$$

Comparing coefficients of numerators on both sides

A = 1

B = 1

C = 2



Hence Inverse of $F(s) = \frac{1}{s+2} + \frac{s+2}{s^2+2s+2} = \frac{1}{s+2} + \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$

Inverse Laplace transform

From table (1), We get

$$f(t) = \left(e^{-2t} + e^{-t}\cos t + e^{-t}\sin t\right)$$

Problem

Find inverse Laplace transform of $\frac{s+2}{s^2-4s+13}$

Answer: $e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$

Problem

Find inverse Laplace transform of

$$\frac{14s+10}{49s^2+28s+13}$$

Answer: $\frac{2}{7}e^{-2t/7}\left(\cos\frac{3}{7}t + \frac{3}{7}t\right)$

Problem

Find the inverse transform of

(i) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$

(ii) $\frac{4s+5}{(s-1)^2(s+2)}$

Answer

(i) $\frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$

Hint: $s^3 - 6s^2 + 11s - 6$ *can be written as* $(s-1)(s-2)(s-3)$. *Method already explained.*

(ii) $\frac{1}{3}e^t + 3te^t - \frac{1}{3}e^{-2t}$



CONVOLUTION THEOREM

If $L^{-1}[F(s)] = f(t)$

and $L^{-1}[G(s)] = g(t)$

Then $L^{-1}[F(s)G(s)] = \int_0^t f(u)g(t-u)du = F*G$

F*G is called the convolution of F and G.

Example

Apply convolution theorem to evaluate

$$L^{-1}\frac{s}{(s^2+a^2)^2}$$

Solution

$$f(t) = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

and $$g(t) = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \frac{1}{a}\sin at$$

By convolution theorem

$$L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \int_0^t \cos au\,\frac{\sin a(t-u)}{a}du \qquad \left[\begin{array}{l} f(u) = \cos au \\ g(t-u) = \frac{1}{a}\sin a(t-u) \end{array}\right.$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au - at)]dt$$

$$= \frac{1}{2a}\left|u\sin at + \frac{1}{2a}\cos(2au - at)\right|_0^t$$

$$= \frac{1}{2a}t\sin at$$

$\therefore$ $$L^{-1}\frac{s}{(s^2+a^2)^2} = \frac{1}{2a}t\sin at$$



Example

Apply convolution theorem to evaluate $L^{-1} = \dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

Solution

$$f(t) = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

and $$g(t) = L^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt$$

By convolution theorem

$$L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right) = \int_0^t \cos au \cos b(t-u)du \qquad \begin{bmatrix} f(u) = \cos au \\ g(t-u) = \cos b(t-u) \end{bmatrix}$$

$$= \frac{1}{2}\left[\cos\{(a-b)u+bt\} + \cos\{(a+b)u-bt)\}\right]du$$

$$= \frac{1}{2}\left|\frac{\sin[(a-b)u+bt}{a-b} + \frac{\sin[(a+b)u-bt}{a+b}\right|_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Example

Find $L^{-1}\left(\dfrac{1}{(s+2)^2(s-2)}\right)$ *using convolution theorem.*

Solution

$$F_1(s) = \frac{1}{(s+2)^2}$$

and $$F_2(s) = \frac{1}{s-2}$$

Now $$f_1(t) = L^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}$$



and $$f_2(t) = L^{-1}\left(\frac{1}{(s-2)}\right) = e^{2t}$$

According to Convolution theorem

$$L^{-1}[F_1(s).F_2(s)] = \int_0^t f_1(x)\, f_2(t-x)dx$$

$$\therefore \quad L^{-1}\left[\frac{1}{(s+2)^2(s-2)}\right] = \int_0^t xe^{-2x}.e^{2(t-x)}dx = \int_0^t xe^{2t-4x}dx$$

$$= \left[x.\frac{e^{2t-4x}}{-4} - \int 1.\frac{e^{(2t-4x)}}{-4}dx\right]_0^t$$

$$= \left[-\frac{x}{4}e^{2t-4x} + \frac{1}{4}\left\{\frac{e^{(2t-4x)}}{-4}\right\}\right]_0^t$$

$$= \frac{-t}{4}e^{2t-4x} - \frac{1}{16}e^{2t-4t} + \frac{1}{16}e^{2t}$$

$$= \frac{-t}{4}e^{-2t} - \frac{1}{16}e^{-2t} + \frac{1}{16}e^{2t}$$

$$= \frac{e^{2t}}{16} - \frac{1}{16}e^{-2t}[4t+1]$$

Example (GTU 2022, 4 marks)

Use convolution theorem to find the inverse Laplace transform $\frac{1}{(s+1)(s^2+1)}$

Solution

Let $$F(s) = \frac{1}{(s+1)(s^2+1)}$$

$$F_1(s) = \frac{1}{s^2+1}; F_2(s) = \frac{1}{s+1}$$

$$f_1(t) = \sin t;\ f_2(t) = e^{-t}$$



By convolution theorem

$$L^{-1}\{F(s)\} = \int_0^t \sin u e^{-(t-u)} du = \int_0^t e^{u-t} \sin u du$$

$$= e^{-t}\left|\frac{e^u}{2}(\sin u - \cos u)\right|_0^t = \frac{e^{-t}}{2}\left[e^t(\sin t - \cos t) + 1\right] = \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t}$$

Example (GTU 2019, 7 marks)

Find the inverse laplace transform of $\frac{1}{s(s+a)^3}$

Solution

Let $F(s) = \frac{1}{s(s+a)^3}$

$$F_1(s) = \frac{1}{(s+a)^3}; F_2(s) = \frac{1}{s}$$

$$f_1(t) = e^{-at}\frac{t^2}{2}; f_2(t) = 1$$

By convolution theorem

$$L^{-1}\{F(s)\} = \int_0^t e^{-au}\frac{u^2}{2} au = \frac{1}{2}\left|\frac{u^2 e^{-au}}{-a} - \frac{2ue^{-au}}{a^2} + \frac{2e^{-au}}{-a^3}\right|_0^t$$

$$= -\frac{1}{2a}t^2e^{-at} - \frac{1}{a^2}te^{-at} - \frac{1}{a^3}e^{-at} + \frac{1}{a^3}$$

Problem

Evaluate using convolution theorem

(i) $L^{-1}\frac{1}{(s^2+1)(s^2+9)}$

(ii) $L^{-1}\frac{s}{(s^2+1)(s^2+4)(s^2+9)}$

Answer:

(i) $\frac{1}{8}(\sin t - \frac{1}{3}\sin 3t)$



(ii) $\frac{1}{12}\cos t - \frac{1}{10}\cos 2t + \frac{1}{60}\cos 3t$

APPLICATION OF LAPLACE TRANSFORMS TO DIFFERENTIALS EQUATIONS

Example

Solve by the method of transforms, the equation

y''' + 2y'' - y' - 2y = 0 given y(0) = y'(0) = 0 and y''(0) = 6.

Solution

Take Laplace transform of both sides

Take the Laplace transform of both sides of the differential equation using the formula

$$L[f^n(t)] = s^nL[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) + ... - f^{n-1}(0)$$

It becomes

$$[s^3\bar{y} - s^2y(0) - sy'(0)] + [2s^2\bar{y} - sy(0) - y'(0)] - [s\bar{y} - y(0)] - 2\bar{y} = 0$$

Apply given values

Using the given conditions, it reduces to

$$(s^3 + 2s^2 - s - 2)\bar{y} = 6$$

$$\therefore \quad \bar{y} = \frac{6}{(s^3 + 2s^2 - s - 2)}$$

Use partial fraction

$$\bar{y} = \frac{6}{(s^3 + 2s^2 - s - 2)} = \frac{1}{(s-1)} - \frac{3}{(s+1)} + \frac{2}{(s+2)}$$ [Method already explained]

Find Laplace transform

$$y = L^{-1}\bar{y} = L^{-1}\frac{1}{(s-1)} - 3L^{-1}\frac{1}{(s+2)} + 2L^{-1}\frac{1}{s+2}$$

$$= e^t - 3e^{-t} + 2e^{-2t}$$

This is required solution.

Example

Use transform method to solve



$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \quad \textit{with } x = 2,\ dx/dt = -1 \textit{ at } t = 0$$

Solution

Take Laplace transform of both sides

Take the Laplace transform of both sides of the differential equation using the formula

$$L[f^n(t)] = s^nL[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) + ... - f^{n-1}(0)$$

It becomes

$$[s^2\bar{x} - sx(0) - x'(0)] - 2[s\bar{x} - x(0)] + \bar{x} = \frac{1}{s-1}$$

Apply given values

Using the given conditions, it reduces to

$$(s^2 - 2s + 1)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 6}{s-1}$$

$$\therefore \qquad \bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

Use partial fraction

$$\bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{2}{(s-1)} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$

Find Laplace transform

$$x = L^{-}(\bar{x}) = 2L^{-1}\frac{1}{s-1} - 3L^{-1}\frac{1}{(s-1)^2} + L^{-1}\frac{1}{(s-1)^3}$$

$$= 2e^t - \frac{3e^t.t}{1!} + \frac{e^tt^2}{2!}$$

$$= 2e^t - 3te^t + \frac{1}{2}t^2e^t$$

Example

Using laplace transformations, find solution of initial value problem

$$y'' - 4y' + 4y = 64\sin 2t$$



Solution

Take Laplace transform of both sides

Take the Laplace transform of both sides of the differential equation using the formula

$$L[f^n(t)] = s^nL[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) + ... - f^{n-1}(0)$$

It becomes

$$[s^2\bar{y} - sy(0) - y'(0)] - 4[s\bar{y} - y(0)] + 4\bar{y} = \frac{64x2}{s^2+4}$$

Apply given values

Using the given conditions, it reduces to

$$s^2\bar{y} - 1 - 4s\bar{y} + 4\bar{y} = \frac{128}{s^2+4}$$

$$\therefore \quad (s^2 - 4s + 4)\bar{y} = 1 + \frac{128}{s^2+4}$$

$$(s-2)^2\bar{y} = 1 + \frac{128}{s^2+4}$$

$$\therefore \quad \bar{y} = \frac{1}{(s-2)^2} + \frac{128}{(s-2)^2(s^2+4)}$$

Use partial fraction

$$\bar{y} = \frac{1}{(s-2)^2} - \frac{8}{s-2} + \frac{16}{(s-2)^2} + \frac{8s}{s^2+4}$$

Find Laplace transform

$$y = L^{-1}\left[\frac{8}{s-2} + \frac{17}{(s-2)^2} + \frac{8s}{s^2+4}\right]$$

$$\therefore \quad y = -8e^{2t} + 17te^{2t} + 8\cos 2t$$

Problem

Using Laplace transforms, find the solution of the initial value problem

$y'' + 9y = 6\cos 3t$

Answer: $2\cos 3t - t\sin 3t$

Problem

Solve using Laplace transform method



$y(0) = -2, y'(0) = 8; y'' + 4y' + 4y = 6e^{-t}$

Answer: $y = 6e^{-t} - 8e^{-2t} - 2te^{-2t}$

Example

Solve the equation by Laplace transform method

$$\frac{dy}{dx} 2y + \int_0^t y dt = \sin t; y(0) = 1$$

Solution

Taking Laplace transform of the given equation

$$[s\bar{y} - y(0)] + 2\bar{y} + \frac{\bar{y}}{s} = \frac{1}{s^2 + 1} \qquad \left[\because L\int_0^t y dt = \frac{\bar{y}}{s}\right]$$

Putting the given values

$$[s\bar{y} - 1] + 2\bar{y} + \frac{\bar{y}}{s} = \frac{1}{s^2 + 1}$$

which finally gives

$$\bar{y} = \frac{s^3 + 2s}{(s+1)^2(s^2+1)}$$

By partial fraction

$$\bar{y} = \frac{1}{s+1} - \frac{3}{2(s+1)^2} + \frac{1}{2(s^2+1)}$$

Taking inverse Laplace transform

$$y = e^{-t} - \frac{3}{2}te^{-t} = \frac{1}{2}\sin t$$

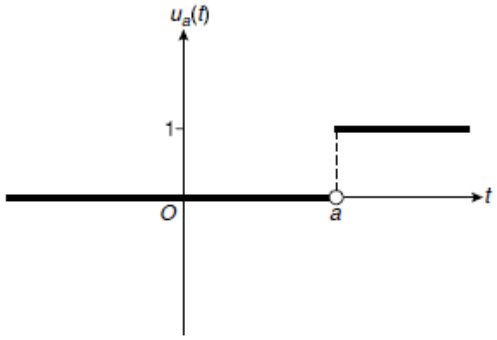
UNIT STEP FUNCTION

Definition

The unit step function u(t- a0 is defined as follows:

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$





It is also denoted as H(t - a).

Transform of unit step function

$$L[u(t-a)] = \int_0^\infty e^{-st}u(t-a)dt = \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.1dt = \frac{e^{-as}}{s}$$

Product of f(t).u(t - a)

It is given as

$$f(t)u(t-a) = \begin{cases} 0 & t < a \\ f(t) & t \geq a \end{cases}$$

The function f(t - a).u(t-a) represents the graph of f(t) shifted through a distance "a" to the right.

Second shifting property

If $L\{f(t)\} = \bar{f}(s)$

Then $L\{f(t-a)u(t-a)\} = e^{-as}\bar{f}(s)$

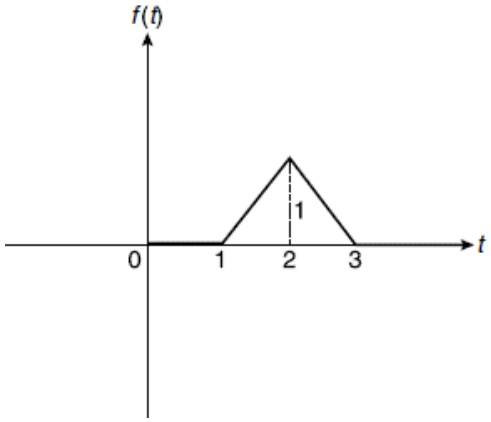
$$= \int_0^\infty e^{-st} f(t-a).u(t-a)dt$$

$$= \int_0^a e^{-st} f(t-a)(0)dt + \int_a^\infty e^{-st} f(t-a)dt$$

$$= \int_0^\infty e^{-s(u+a)} f(u)du = e^{-sa}\int_0^\infty e^{-su} f(u)du = e^{-as}\bar{f}(s)$$

Example

Express the function shown in figure in terms of unit step function and find its Laplace transform.



Solution

Given that (from the figure)

$$f(t)=\begin{cases} t-1 & 1<t<2 \\ 3-t & 2<t<3 \end{cases}$$

$$f(t)=(t-1)\{u(t-1)-u(t-2)+(3-t)\{u(t-2)-u(t-3)\}$$

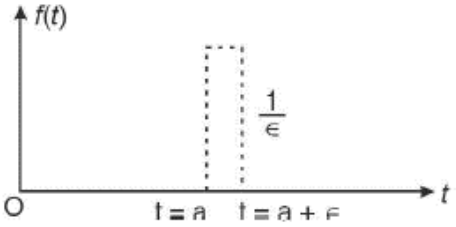
$$= (t-1)u(t-1)-2(t-2)u(t-2)+(t-3)(t-3)$$

As $L\{f(t-a)u(t-a)\}=e^{-as}\,\overline{f}(s)$

$\therefore$ $L\{f(t)\}=e^{-s}\;L\{f(t)\}=e^{-s}.\frac{1}{s^2}-2e^{-2s}.\frac{1}{s^2}+e^{-3s}.\frac{1}{s^2}=\frac{e^{-s}(1-e^{-s})^2}{s^2}$ [$\because$ f(t) = t]

UNIT IMPULSE FUNCTION (DIRACT DELTA FUNCTION)

It frequently occurs in many problems in electrical engineering, physics and mechanical engineering that a *large force acts for a very short duration*. To deal with such problems, we define unit impulse function or Dirac-delta function.



$$\delta(t-a)=\begin{cases} \infty & t=a \\ 0 & t\neq a \end{cases}$$

and $\int_0^\infty \delta(t-a)dt=1$

Laplace Transform of unit impulse function

$$\int_0^\infty f(t)\delta(t-a)dt = \int_a^{a+\varepsilon} f(t).\frac{1}{\varepsilon}dt$$

$$= f(a)$$

If *If* $f(t) = e^{-st}; L[\delta(t-a)] = e^{-as}$

Also $$\int_{-\infty}^\infty f(t)\delta'(t-a)dt = -f'(a)$$

Example

Evaluate $\int_{-\infty}^\infty e^{-5t}\delta(t-2)dt$

Solution

$$\int_{-\infty}^\infty e^{-5t}\delta(t-2)dt = e^{-5x2} = e^{-10}$$

Example

Find Laplace transform of $t^3\delta(t-4)$.

Solution

$$Lt^3\delta(t-4) = \int_0^\infty e^{-st}t^3\delta(t-4)dt = 4^3e^{-4s}$$

Example

Evaluate $\int_0^\infty \sin 2t\delta(t-\pi/4)dt$

Solution

From $$\int_0^\infty f(t)\delta(t-a)dt = f(a)$$

$$\int_0^\infty \sin 2t\delta(t-\pi/4)dt = \sin\left(2.\frac{\pi}{4}\right) = 1$$



ASSIGNMENT

Q.1. (AKTU 2018, 2020, 2021, 2 marks): What do you mean by initial value problem?

Q.2. (PTU 2018, 2020, 2 marks): State the change of scale property of Laplace transforms.

Q.3. (PTU 2020, 2 marks): State and prove Second shifting property of Laplace transform.

Q.4. (GTU 2022, 7 marks): Solve the initial value problem using Laplace transform.

$y'' + y' = t^2 + 2t$, $y(0) = 4$, $y'(0) = -2$

Q.5. (PTU 2019, 2 marks): Find $L(t^2e^{-2t}\cos t)$

Answer: Solved in this module.

Q.6. (PTU 2019, 2 marks): Find $L\{(te^{-t}\sin 4t)\}$

Answer: $\dfrac{2(s+4)}{\left(s^2+8s+17\right)^2}$

Q.7. (PTU 2020, 2 marks): Evaluate $L(\cos^2\alpha \sin\beta t)$.

Q.8. (PTU 2020, 5 marks): Using Laplace evaluate $\int_0^\infty t^3e^{-t}\sin t dt$

Answer: Solved in this module.

Q.9. (PTU 2020, 5 marks): Find inverse Laplace of $\dfrac{s^2}{(s^2+a^2)^2}$

Answer: (1/2a)(sinat + atcosat)

Q.10. (GTU 2019, 4 marks): Solve

(a) $L\left\{e^t(\sin 3t + t^3)\right\}$

(b) $L^{-1}\left\{\dfrac{6s-7}{s^2+5}\right\}$

Q.11. (GTU 2019, 7 marks): State the convolution theorem and apply it to evaluate $L^{-1}\left[\dfrac{1}{s(s+a)^3}\right]$

Answer: Solved in this module.

Q.12. (GTU 2020, 7 marks): State the convolution theorem and apply it to evaluate $L^{-1}\left[\dfrac{1}{(s^2+a)^2}\right]$

Q.13. (GTU 2021, 7 marks): Find the inverse Laplace transform of $\dfrac{1}{(s^2+4)^2}$ by Convolution theorem.

Answer: $\dfrac{1}{16}(\sin 2t - 2t\cos 2t)$



Q.14. (GTU 2022, 4 marks): Use convolution theorem to find the inverse Laplace transform of $\frac{1}{(s+1)(s^2+1)}$.

Answer: Solved in this modue.

Q.15. (GTU 2019, 3 marks): Find the Laplace transform of $f(t)=\begin{Bmatrix}0 & 0\le t<2\\ 3 & t\ge 2\end{Bmatrix}$

Q.16. (GTU 2022, 4 marks): Find the Laplace transform of $f(t)=\begin{Bmatrix}t^2 & 9<t<1\\ 4t & t>1\end{Bmatrix}$

Q.17. (KU 2019, 7.5 marks): Find the Laplace transform of the function

$$f(t)=\begin{Bmatrix}\sin\omega t & 0<t<\pi/\omega\\ 0 & \pi/\omega<t<2\pi/\omega\end{Bmatrix}$$

Answer: Solved in this module.

Q.18. (GTU 2019, 3 marks): Find the inverse Laplace transform of $\frac{6e^{-2s}}{(s^2+4)}$

Q.19. (GTU 2019, 4 marks): Find

(i) $L\left[\int_0^t e^{-t}\cos u\,du\right]$

(ii) $L^{-1}\left[\frac{2s+2}{s^2+2s+10}\right]$

Q.20. (GTU 2019, 2021, 4 marks): Find

(i) $L\left[\int_0^t\int_0^t \sin au.du\,du\right]$

(ii) $L^{-1}\left[\frac{s+2}{s^2+4s+8}\right]$

Q.21. (GTU 2021, 4 marks): Find $L^{-1}\left[\frac{6+s}{s^2+6s+13}\right]$

Answer: $\frac{1}{2}e^{-3t}(2\cos 2t+3\sin 2t)$

Q.22. (GTU 2022, 3 marks): Find $L^{-1}\left[\frac{3s+4}{s^2+9}\right]$

Answer: 3cos3t + (4/3)sin3t



Q.23. (GTU 2019, 4 marks): Find $L^{-1}\left[\frac{e^{-2s}}{s^2}\right]$

Answer: $L^{-1}(1/s^2) = t$

$$L^{-1}\left(e^{-2s}/s^2\right) = (t-2)u_2(t) = \begin{bmatrix} 0 & 0 \le t < 2 \\ t-2 & t \ge 2 \end{bmatrix}$$

Q.24. (GTU 2020, 3 marks): Find $L\left[e^{-t}\sin^2 t\right]$

Q.25. (GTU 2020, 4 marks): Find $L^{-1}\left[\frac{3}{s^2+6s+18}\right]$

Q.26. (GTU 2021, 4 marks): Find $L^{-1}\left[\frac{2s+1}{s(s+1)}\right]$

Q.27. (GTU 2022, 3 marks): Find the inverse Laplace transform of $\frac{e^{-\pi s}}{s^2-2s+2}$

Q.28. (GTU 2022, 4 marks): Find the Laplace transform of $(t + 1)^2 . e^t$

Q.29. (GTU 2022, 3 marks): Find the Laplace transform of $\int_0^t e^{-2t}t^3 dt$

Q.30. (KU 2019, 7.5 marks): Find the Laplace transform of the function $e^{-t}\cos^2 t$.

Q.31. (KU 2019, 7.5 marks): Find the inverse Laplace transform of the function $\frac{s}{s^2+6s+25}$

Q.32. (GTU 2019, 7 marks): Using Laplace Transformation, solve y'' + 6y = 1, y(0) = 2, y'(0) = 0

Q.33. (GTU 2020, 7 marks): Solve y'' - y' - 2y = 0 with y(0) = 1, y'(0) = 0 using Laplace transform.

Q.34. (GTU 2021, 7 marks): Solve by Laplace transform $y' + 2y = e^{-3t}$ with y(0) = 1.

Q.35. (KU 2019, 7.5 marks): Solve the following differential equation using Laplace transform

$$\frac{d^2y}{dt^2} + 25\frac{dy}{dx} = 10\cos 5t$$

where y(0) = 2 and y'(0) = 0.

Q.36. (PTU 2018, 5 marks): Using Laplace transforms, solve the differential equation

$$\frac{d^2x}{dt^2} + 9x = \cos 2t$$

where x(0) = 1 and $x\left(\frac{\pi}{2}\right) = -1$.



Answer: $y(t) = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t$

