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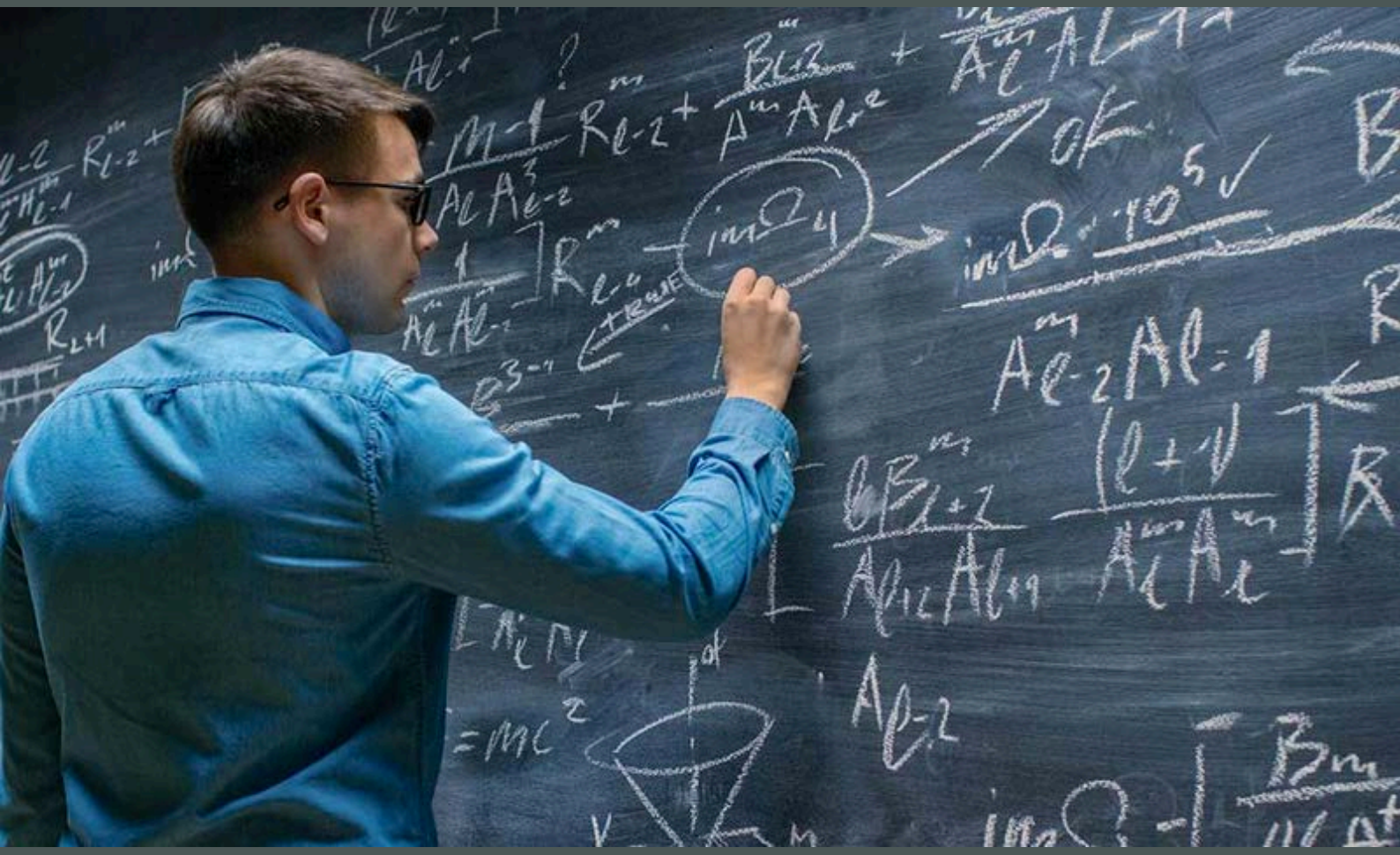

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FOURIER SERIES, FOURIER INTEGRALS AND FOURIER TRANSFORMS

FOURIER SERIES

Fourier series is used in the analysis of periodic functions. Many of the phenomena studied in engineering and sciences are periodic in nature, e.g., current and voltage in an ac circuit.

CONVERGENCE OF THE FOURIER SERIES (DIRICHLET'S CONDITIONS)

A function f (x) can be represented by a complete set of orthogonal functions within the interval **(c, c + 2l)**. The Fourier series of the function f (x) exists only if the following conditions are satisfied:

1. f (x) is periodic, i.e., **f (x) = f (x + 2l)**, where 2l is the period of the function f (x).
2. f (x) and its integrals are **finite and single-valued**.
3. f (x) has a **finite number of discontinuities**, i.e., f (x) is piecewise continuous in the interval (c, c + 2l).
4. f (x) has a **finite number of maxima and minima**.

These conditions are known as **Dirichlet's conditions**.

TRIGONOMETRIC FOURIER SERIES (OR FOURIER SERIES)

Hence, any function f (x) can be represented in terms of these orthogonal functions in the interval (c, c + 2l) for any value of c.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

This series is known as a trigonometric Fourier series or simply, a Fourier series.

Here $$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}dx$$



and $$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx$$

When c = 0 and 2l = 2π

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx$$

When c = -π and 2l = 2π

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$$

When c = 0

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$$

When c = -l

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$



IMPORTANT FORMULAS

- $\int_0^{2\pi} \sin nx dx = 0$
- $\int_0^{2\pi} \cos nx dx = 0$
- $\int_0^{2\pi} \sin^2 nx dx = \pi$
- $\int_0^{2\pi} \cos^2 nx dx = \pi$
- $\int_0^{2\pi} \sin nx . \sin mx dx = 0$
- $\int_0^{2\pi} \cos nx . \cos mx dx = 0$
- $\int_0^{2\pi} \sin nx . \cos mx dx = 0$
- $\int_0^{2\pi} \sin nx . \cos nx dx = 0$
- $\sin 2n\pi = \sin 0 = 0$
- $\sin n\pi = 0$
- $\cos 2n\pi = \cos 0 = 1$
- $\cos(-n\pi) = \cos(n\pi)$ and $\cos n\pi = (-1)^n$
- $\int_{-a}^{a} f(x) = 2\int_0^a f(x), if\ \ f(x)\ is\ even$
  $= 0\ if\ \ f(x)\ is\ odd$
- $\int u\, v\, dx = u\int v\, dx - \int u' (\int v\, dx)\, dx$ (integration by parts)
- $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
- $\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$

Example (PTU 2018, GTU 2014)

Obtain the Fourier series of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ *in the interval* $0 \le x \le 2\pi$.

Hence deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$

Solution

We will use following Fourier series for period 2π



$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 dx$$

$$= = \frac{1}{8\pi}\left|\frac{(\pi - x)^3}{-3}\right|_0^{2\pi} = -\frac{1}{24\pi}\left(-\pi^3 - \pi^3\right) = \frac{\pi^2}{12}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \cos nxdx$$

$$= \frac{1}{4\pi}\left|(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[2\pi.\frac{\cos 2n\pi}{n^2} - \left(-2\pi\frac{\cos 0}{n^2}\right)\right] \qquad \text{[As } \sin 2n\pi = \sin 0 = 0\text{]}$$

$$= \frac{1}{4\pi}\left(\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right) \qquad \text{[As } \cos 2n\pi = \cos 0 = 1\text{]}$$

$$= 1/n^2$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nxdx$$

$$= \frac{1}{4\pi}\left|(\pi - x^2)\left(-\frac{\cos nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi}$$

$$= 0$$

$$\therefore \quad f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + ..$$

Put x = π, we get $0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + ..$

$$\therefore \quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2 - ...}$$



Problem (GTU 2013)

Obtain the Fourier series of $f(x)=\frac{1}{2}(\pi-x)$ *in the interval* $0 \le x \le 2\pi$.

Hence deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$

Answer: $f(x)=\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$

Problem (GTU 2014, 2019, 7 marks)

Find the Fourier series of f(x) = x + |x| in the interval $-\pi < x < \pi$.

Solution

We will use following Fourier series for period 2π

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}(x+|x|)dx=\frac{1}{2\pi}\left[\int_{-\pi}^{\pi}xdx+\int_{-\pi}^{\pi}|x|dx\right]$$

$$=\frac{1}{2\pi}\left[0+2\int_0^{\pi}|x|dx\right] \quad \left[\because \int_{-a}^{a}f(x)=2\int_0^{a}f(x), if\ f(x)\ is\ even \;\; =0\ if\ f(x)\ is\ odd\right]$$

$$=\frac{1}{\pi}\int_0^{\pi}xdx=\frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}=\frac{1}{\pi}\left(\frac{\pi^2}{2}\right)=\frac{\pi}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+|x|)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}x\cos nxdx+\int_{-\pi}^{\pi}|x|\cos nxdx\right]$$

$$=\frac{1}{\pi}\left[0+2\int_0^{\pi}|x|\cos nxdx\right] \quad \left[\because x\cos nx\ is\ odd\ and\ |x|\cos nx\ is\ even\ function\right]$$

$$=\frac{2}{\pi}\int_0^{\pi}x\cos nxdx=\frac{2}{\pi}\left|x\frac{\sin nx}{n}-(1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} \quad [\sin n\pi = \sin 0 = 0]$$

$$=\frac{2}{\pi}\left|x\frac{\sin nx}{n}-(-1)\frac{\cos 0}{n^2}\right|_0^{\pi}=\frac{2}{\pi n^2}\left[(-1)^n-1\right] \quad [\cos n\pi = (-1)^n \text{ and } \cos 0 = 0)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+|x|\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}x\sin nxdx+\int_{-\pi}^{\pi}|x|\sin nxdx\right]$$



$$= \frac{1}{\pi}\left[2\int_0^{\pi} x\sin nx\,dx = 0\right] \quad \left[\because x\sin nx\ is\ even, |x|\sin x\ is\ odd\ \ function\right]$$

which leads to

$$= \frac{2}{\pi}\left|x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi} = -\frac{2}{n}(-1)^n$$

using [sinnπ = sin 0 = 0 and cosnπ = $(-1)^n$.

$$\therefore \qquad f(x) = \frac{\pi}{2}+\frac{2}{\pi}\left[-\frac{2}{1^2}\cos x-\frac{2}{3^2}\cos 3x-\frac{2}{5^2}\cos 5x-....\right]$$

$$x+|x| = \frac{\pi}{2}+\frac{2}{\pi}\left[-\frac{2}{1^2}\cos x-\frac{2}{3^2}\cos 3x-\frac{2}{5^2}\cos 5x-...\right]$$

$$-2\left[-\frac{1}{1}\sin x = \frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+..\right]$$

$$= \frac{\pi}{2}-\frac{4}{\pi}\left(\frac{1}{1^2\cos x}+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+...\right)$$
$$+2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-...\right)$$

Problem

Find the Fourier series of f(x) = x in the interval (0,2π).

Answer: $f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{1}{n}\sin nx;\ x = \pi-2\left(\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+...\right)$

Example (GTU 2012)

Find the Fourier series of f(x) = x + x^2 in the interval (-π, π) and hence deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-...$$

Solution

The Fourier series f(x) with period 2π is given by

$$f(x) = a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x+x^2)dx$$

$$= \frac{1}{2\pi}\left|\frac{x^2}{2}+\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nxdx$$

$$= \frac{1}{\pi}\left|(x+x^2)\left(\frac{\sin nx}{n}\right)-(1+2x)\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{4(-1)^n}{n^2} \quad \text{using } [\cos(-n\pi) = \cos(n\pi) \text{ and } \cos n\pi = (-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nxdx$$

$$= \frac{1}{\pi}\left|(x+x^2)\left(-\frac{\cos nx}{n}\right)-(1+2x)\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{-2(-1)^n}{n} \text{ using } \cos(-n\pi) = \cos n\pi \text{ and } \cos n\pi = (-1)^n$$

Hence $$f(x) = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}-2\sum_{n}^{(-1)^n}\sin nx$$

$$\therefore \quad x+x^2 = \frac{\pi^2}{3}+4\left(-\frac{1}{1^2}\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+\ldots\right)$$

$$-2\left(-\frac{1}{1}\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\ldots\right)$$

Putting x = 0 in this equation, we get

$$\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots$$

Example

Example (GTU 2013, 2021, 7 marks)

Find the Fourier series of



$$f(x)=\begin{cases}0 & -\pi<x<0\\ x & 0<x<\pi\end{cases}$$

Solution

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}0dx+\int_{0}^{\pi}xdx\right]=\frac{\pi}{4}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{0}^{\pi}x\cos nxdx=\frac{1}{\pi}\left|x\frac{\sin nx}{n}-(1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]$$

and $$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{0}^{\pi}x\sin xdx$$

$$=\frac{1}{\pi}\left|x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}=-\frac{(-1)^n}{n}$$

$\therefore$ $$f(x)=\frac{\pi}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]\cos nx-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$=\frac{\pi}{4}+\frac{1}{\pi}\left(-\frac{2}{1^2}\cos x-\frac{2}{3^2}\cos 3x-\frac{2}{5^2}\cos 5x-\ldots\right)$$
$$-\left(-\frac{1}{1}\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\ldots\right)$$

Problem (GTU 2014)

Find the Fourier series of

$$f(x)=-\pi \quad -\pi<x<0$$
$$=x \quad 9<x<\pi$$

Hence deduce that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

Answer:

Answer: $f(x)=-\frac{\pi}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos nx+\sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]\sin nx$



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Put x = 0, we get $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Example

Find the Fourier series of $f(x) = \begin{cases} x & -\pi/2 < x < \pi/2 \\ \pi - x & \pi/2 < x < 3\pi/2 \end{cases}$

Solution

The Fourier series f(x) with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Now $$a_0 = \frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi/2}^{\pi/2} x dx + \int_{\pi/2}^{3\pi/2} (\pi - x)dx\right]$$

$$= \frac{1}{2\pi}\left[\left|\frac{x^2}{2}\right|_{-\pi/2}^{\pi/2} + \left|\pi x - \frac{x^2}{2}\right|_{\pi/2}^{3\pi/2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\cos nx dx = \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} x\cos nx dx + \int_{\pi/2}^{3\pi/2} (\pi - x)\cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\pi/2}^{\pi/2} + \left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\pi/2}^{3\pi/2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin n\pi\sin\frac{n\pi}{2}\right]$$

= 0 (by using various trignometric formulae given earlier)

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\sin nx dx = \frac{1}{\pi}\left[\int_{-\pi/2}^{3\pi/2} (\pi - x)\sin nx dx + \int_{\pi/2}^{3\pi/2} (\pi - x)\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\left|x\left(\frac{-\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\pi/2}^{\pi/2} + \left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{\pi/2}^{3\pi/2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{3}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{3n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$



$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2}-\cos\frac{n\pi}{2}\right)+\frac{3}{n^2}\sin\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\sin\frac{n\pi}{2}+\frac{3}{n^2}\sin\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi n^2}\left[3\sin\frac{n\pi}{2}-\sin\frac{3n\pi}{2}\right]$$ (by using various trignometric formulae given earlier)

$$\therefore \quad f(x)=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[3\sin\frac{n\pi}{2}-\sin\frac{3n\pi}{2}\right]\sin nx$$

FOURIER SERIES OF EVEN AND ODD FUNCTIONS

A function f (x) is said to be **even** if f (x) = f (x) and **odd** if f (x) = f (x) for all x.

Properties of Even and Odd Functions

- The product of two even functions is even.
- The product of two odd functions is even.
- The product of an even function and an odd function is odd.
- The sum or difference of two even functions is even.
- The sum or difference of two odd functions is odd.
- If f (x) is even, $\int_{-l}^{l} f(x)dx = 2\int_{0}^{l} f(x)dx$
- If f (x) is odd, $\int_{-l}^{l} f(x)dx = 0$

When f (x) is an even function

$$a_0 = \frac{1}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$b_n = 0$

When f (x) is an odd function

$$a_0 = 0$$

$$a_n = 0$$



$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

We see that the Fourier series of an **even** function consists entirely of **cosine** terms while the Fourier series of an **odd** function consists entirely of **sine** terms.

Example (GTU 2020, 4 marks)

Find Fourier series for $f(x) = x^2; -\pi \le x \le \pi$

Solution

$f(x) = x^2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an **even function** is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{\pi} = \frac{\pi^2}{3}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx dx$$

$$= \frac{2}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

$\therefore$ $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$\therefore$ $$x^2 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + ...\right)$$

Example

Find the Fourier series of $f(x) = x^3$ *in the interval* $(-\pi, \pi)$.

Solution

$f(x) = x^3$ is an odd function.



Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx dx$$

$$= \frac{2}{\pi}\left| x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\pi^3.\frac{\cos n\pi}{n} + 6\pi\frac{\cos n\pi}{n^3}\right) \qquad [\sin n\pi = 0]$$

$$= 2(-1)^n\left(-\frac{\pi^2}{n} + \frac{6}{n^3}\right) \qquad [\cos n\pi = (-1)^n$$

Problem

Find the Fourier series of f(x) = |x| in the interval [-π, π].

Answer: $f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2\cos 5x + \ldots}\right)$$

Example (PTU 2019, MDU 2010, AKTU 2008, UTU 2007)

Find the Fourier series of f(x) = xsinx in the interval (-π, π).

Solution

f (–x) = –x sin (–x) = x sin x = f (x)

f (x) = x sin x is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x\sin x dx$$

$$= \frac{1}{\pi}\left|x(-\cos x) - (-\sin x)\right|_0^{\pi} = 1$$



$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]dx$$

$$= \frac{1}{\pi}\left|x\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - \left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos(n+1)\pi}{n+1} + \pi.\frac{\cos(n-1)\pi}{n-1}\right] \qquad [\sin(n+1)\pi = \sin(n-1)\pi = 0]$$

$$= \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} = \frac{-2(-1)^n}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1} \qquad [\cos(n+1)\pi = \cos(n-1)\pi = (-1)^n]$$

For n = 1

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx = \frac{1}{\pi}\left|-x\frac{\cos 2x}{2} + \frac{\sin 2x}{4}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(-\pi.\frac{\cos 2\pi}{2}\right) = -\frac{1}{2} \qquad [\sin 2\pi = 0; \cos 2\pi = 1]$$

$$\therefore \qquad f(x) = 1 - \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

$$\therefore \qquad x\sin x = \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

$$= \frac{1}{2} - 2\left(\frac{1}{3}\cos 2x - \frac{1}{8}\cos 3x + \frac{1}{15}\cos 4x - \ldots\right)$$

Problem

Find the Fourier series of $f(x) = 1 - x^2$ in the interval (-1, 1).

Answer: $f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$

$$1 - x^2 = \frac{2}{3} - \frac{4}{\pi^2}\left(-\frac{1}{1^2}\cos \pi x + \frac{1}{2^2}\cos 2\pi x - \frac{1}{3^2}\cos 3\pi x + \ldots\right)$$

FOURIER INTEGRAL



We know that the Fourier series of the function f(x) in any interval (–l, l) is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \frac{n\pi x}{l}$$

where $a_0 = \frac{1}{2l}\int_{-l}^{l} f(t)dt$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}dt$$

and $b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}dt$

Putting these values in above equation of Fourier series, we finally get

$$f(x) = \frac{1}{\pi}\left[\int_0^{\infty} f(t)\cos\omega(t-x)dt\right]d\omega$$

This equation is called the **Fourier integral** of f (x).

Expanding this equation, we finally get

$$f(x) = \int_0^{\infty} A(\omega)\cos\omega x d\omega + \int_0^{\infty} B(\omega)\sin\omega x d\omega$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t dt$

and $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t dt$

Example (GTU 2022, 7 marks)

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

Solution

f(x) is an even function. The Fourier cosine integral of f (x) is given by

$$f(x) = \int_0^{\infty} A(\omega)\cos\omega x d\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(t)\cos\omega t dt = \frac{2}{\pi}\int_0^{\infty} (1-t^2)\cos\omega t dt$$



$$= \frac{2}{\pi}\left|(1-t^2)\left(\frac{\sin\omega t}{\omega}\right)-(-2t)\left(-\frac{\cos\omega t}{\omega^2}\right)+(-2)\left(\frac{\sin\omega t}{\omega^3}\right)\right|_0^t$$

$$= \frac{2}{\pi}\left(-\frac{2\cos\omega}{\omega^2}+\frac{2\sin\omega}{\omega^3}\right)=\frac{4}{\pi}\left(\frac{\sin\omega-\omega\cos\omega}{\omega^3}\right)$$

$\therefore$ $$f(x)=\frac{4}{\pi}\int_0^\infty \frac{\sin\omega-\omega\cos\omega}{\omega^3}\cos\omega x d\omega$$

Example (GTU 2013, 2014, AKTU 2018, 2020, 7 marks)

Express the function $f(x)=\begin{Bmatrix}1 & |x|\le 1\\ 0 & |x|<1\end{Bmatrix}$ *as Fourier integral.*

Hence evaluate $\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}d\omega$

Solution

The function f(x) is an even function. The Fourier cosine integral of f(x) is given by

$$f(x)=\int_0^\infty A(\omega)\cos\omega x d\omega$$

$$A(\omega)=\frac{2}{\pi}\int_0^\infty f(t)\cos\omega t dt=\frac{2}{\pi}\int_0^1 1.\cos\omega t dt=\frac{2}{\pi}\left|\frac{\sin\omega t}{\omega}\right|_0^1=\frac{2}{\pi}\frac{\sin\omega}{\omega}$$

$\therefore$ $$f(x)=\frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}d\omega$$

Now $$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}d\omega=\frac{\pi}{2}f(x)=\begin{cases}\pi/2 & |x|<1\\ 0 & |x|>1\end{cases} \quad (1)$$

At |x| = 1, i.e. x = ± 1, f(x) is discontinuous.

At x = 1

$$f(x)=\frac{1}{2}\left[\lim_{x\to 1^-}f(x)+\lim_{x\to 1^+}f(x)\right]=\frac{1}{2}(1+0)=\frac{1}{2}$$

At x = -1

$$f(x)=\frac{1}{2}\left[\lim_{x\to 1^-}f(x)+\lim_{x\to 1^+}f(x)\right]=\frac{1}{2}(0+1)=\frac{1}{2}$$

Hence, from eq. (1)



$$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}d\omega = \begin{cases} \pi/2 & |x|<1 \\ \pi/4 & |x|=1 \\ 0 & |x|>1 \end{cases}$$

FOURIER TRANSFORM

FOURIER TRANSFORM FORMULA

Fourier transform	$F\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)dx$
Inverse Fourier transform	$F^{-1}\{F(\omega)\} = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} F(\omega)d\omega$
Fourier complex transform	$F\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx$
Inverse Fourier complex transform	$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega$
Fourier sine transform	$F_s[f(x)] = F(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin\omega x dx$
Inverse Fourier sine transform	$F(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(\omega)\sin\omega x d\omega$
Fourier cosine transform	$F_c[f(x)] = F(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\omega x dx$
Inverse Fourier cosine transform	$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\omega x dx$
Fourier complex integral	$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}du\int_{-\infty}^{\infty} f(t)e^{-iut}dt$

PROPERTIES OF FOURIER TRANSFORM

Linear property

If $F_1(\omega)$ and $F_2(\omega)$ are Fourier transforms of $f_1(x)$ and $f_2(x)$ respectively, then

$$F[af_1(x)+bf_2(x)] = aF_1(\omega)+bF_2(\omega)$$



where a and b are constants.

Change of scale property

if F(ω) is the complex Fourier transform of f(x), then

$$F\{f(ax)\} = \frac{1}{a}F\left(\frac{\omega}{a}\right)$$

Shifting property

If F(ω) is the complex Fourier transform of f(x), then

$$F\{f(x-a)\} = e^{i\omega a}F(\omega)$$

Modulation theorem

If F(ω) is the complex Fourier transform of f(x), then

$$f\{f(x).\cos ax\} = \frac{1}{2}[F(\omega+a)+F(\omega-a)]$$

CONVOLUTION THEOREM OF FOURIER TRANSFORM

The Fourier transform of the convolution of f(x) and g(x) is the product of their Fourier transform.

$$F\{f(x)*g(x)\} = F\{f(x)\}.F\{g(x)\}$$

Properties

- $F_s[af(x)+bg(x)] = aF_s\{f(x)\}+bF_s\{g(x)\}$
- $F_c[af(x)+bg(x)] = aF_c\{f(x)\}+bF_c\{g(x)\}$
- $F_s\{f(ax)\} = \frac{1}{a}F_s\left(\frac{\omega}{a}\right)$
- $F_c\{f(ax)\} = \frac{1}{a}F_c\left(\frac{\omega}{a}\right)$
- $F\{f(x-a)\} = e^{i\omega a}F(\omega)$
- $F\{f(x+a)\} = e^{-i\omega a}F(\omega)$
- $F\{e^{iax}f(x)\} = F(\omega+a)$
- $F\{e^{-iax}f(x)\} = F(\omega-a)$



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- $F\{f(x)\cos at\} = \frac{1}{2}\{F(\omega + a) + F(\omega - a)\}$
- $F\{x^n f(x)\} = (-1)^n \frac{d^n}{d\omega^n} F(\omega)$
- $F\{x' f(x)\} = -i\omega F(\omega)$
- $F\left\{\int_a^x f(x)dx\right\} = \frac{F(\omega)}{-i\omega}$
- $\int_0^\infty e^{-ax} \sin \omega x dx = \frac{\omega}{\omega^2 + a^2}$
- $\int_0^\infty e^{-ax} \cos \omega x dx = \frac{a}{\omega^2 + a^2}$

Example

Find the Fourier transform of $e^{-a|x|}, a > 0$

Solution

Let $f(x) = e^{-a|x|}$

Then $$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos \omega x + i \sin \omega x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \cos \omega x dx \qquad \left[\left\{\int_{-\infty}^{\infty} e^{-a|x|} \sin \omega x dx = 0\right\} \textit{being odd function}\right.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-a|x|} \cos \omega x dx = \frac{2}{\sqrt{2\pi}} \left[\frac{e^{-a|x|}}{a^2 + \omega^2} (-a \cos \omega x + \omega \sin \omega x)\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{2\pi}} \left[0 - \frac{e^{-0}(-a)}{a^2 + \omega^2}\right] = \frac{2}{\sqrt{2\pi}} \left(\frac{a}{a^2 + \omega^2}\right) \qquad [e^{-\infty} = 0, \sin 0 = 0, \cos 0 = 0]$$

Example

Find the Fourier transform of

$$f(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$



Solution

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)(\cos\omega x + i\sin\omega x)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\underset{\text{even function}}{(1-|x|)\cos\omega x} + \frac{i}{\sqrt{2\pi}}\int_{-1}^{1}\underset{\text{odd function}}{(1-|x|)\sin\omega x dx}$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{1}(1-x)\cos\omega x dx + 0 = \sqrt{\frac{2}{\pi}}\left[\left\{\frac{(1-x)\sin\omega x}{\omega}\right\}_0^1 + \int_0^1 \frac{\sin\omega x}{\omega}dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 + \left\{\frac{-\cos\omega x}{\omega^2}\right\}_0^1\right] = \sqrt{\frac{2}{\pi}\left(\frac{1-\cos\omega}{\omega^2}\right)}$$

Example

Find the Fourier transform of

$$F(x) = \begin{cases} 1 & |x|<1 \\ 0 & |x|>1 \end{cases}$$

Hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$

Solution

By the definition of Fourier transform, we have

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx$$

$$= \int_{-\infty}^{-1} f(x)e^{i\omega x}dx + \int_{-1}^{1} f(x)e^{i\omega x}dx + \int_{1}^{\infty} f(x)e^{i\omega x}dx$$

By the given definition of f(x), we have

$$F\{f(x)\} = \int_{-\infty}^{-1} 0.e^{i\omega x}dx + \int_{-1}^{1} 1.e^{i\omega x}dx + \int_{1}^{\infty} 0.e^{i\omega x}dx$$

$$= \int_{-1}^{1} e^{i\omega x}dx = \left[\frac{e^{i\omega x}}{i\omega}\right]_{-1}^{1} = \left[\frac{e^{i\omega}}{i\omega} - \frac{e^{-i\omega}}{i\omega}\right] = \frac{2}{\omega}\sin\omega$$

By the inverse Fourier transform, we have



$$F^{-1}\{f(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{-i\omega x}d\omega = f(x)$$

$$\therefore \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\left(\frac{2}{\omega}\sin\omega\right)d\omega$$

$$\therefore \quad f(x) = \begin{cases} 1 & |x|<1 \\ 0 & |x|>1 \end{cases} = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\left(\frac{\sin\omega}{\omega}\right)$$

Putting x = 0, we get

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\omega}{\omega}\right)d\omega = 1$$

$$\therefore \quad \int_{-\infty}^{\infty}\left(\frac{\sin\omega}{\omega}\right)d\omega = \pi$$

Since the integran namely (sinω/ω) is an even function in the interval (-∞, ∞),

We have $\int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$

Example (JNTU 2002, 2004)

Find the Fourier sine transform of $f(x) = e^{-ax}$, $a > 0$.

Hence deduce that $\int_0^{\infty}\frac{x\sin x}{1+x^2}dx = \frac{\pi}{2}e^{-a}$

Solution

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin\omega x dx$$

$$\therefore \quad F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin\omega x dx = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2}\right)$$

By inversing sine transform, we get

$$F(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(\omega)\sin\omega x d\omega = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2}\right)\sin\omega x d\omega$$

$$\therefore \quad \frac{\pi}{2}f(x) = \int_0^{\infty}\frac{\omega\sin\omega x}{\omega^2+a^2}d\omega$$

$$\frac{\pi}{2}f(e^{-ax}) = \int_0^{\infty}\frac{\omega\sin\omega x}{\omega^2+a^2}d\omega$$



Put a = 1, x = a, we get

$$\frac{\pi}{2}e^{-x} = \int_0^\infty \frac{\omega \sin \omega a}{\omega^2 + 1} d\omega$$

Replace "ω" by "x".

$$\int_0^\infty \frac{x \sin xa}{1 + x^2} dx = \frac{\pi}{2} e^{-a}$$

Example (JNTU 2002, 2004)

Find the Fourier cosine transform of $f(x) = e^{-ax}$, $a>0$.

Hence deduce that $\int_0^\infty \frac{\cos \lambda x}{x^2 + a} dx,\ a > 0$

Solution

The Fourier cosine transform is

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos \lambda x dx = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{(a^2 + \lambda^2)} (-a \cos \lambda x + \lambda \sin \lambda x) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + \lambda^2}$$

By inverse Fourier cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\lambda) \cos \lambda x dx$$

$$\therefore \quad e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + \lambda^2} \cos \lambda x dx = \frac{2a}{\pi} \int_0^\infty \frac{1}{a^2 + \lambda^2} \cos \lambda x dx$$

$$\therefore \quad \frac{\pi}{2a} e^{-ax} = \int_0^\infty \frac{\cos \lambda x}{a^2 + \lambda^2} dx$$

Interchanging x by λ

$$\int_0^\infty \frac{\cos \lambda x}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-a\lambda}$$

Problem

Find the Fourier sine transform of e^{-3x}.



Answer: $\sqrt{\frac{2}{\pi}\left(\frac{\omega}{\omega^2+9}\right)}$

Example

Find Fourier sine transform of $f(t) = te^{-at}$, $a > 0$.

Solution

We have $F_s(\omega) = \int te^{-at} \sin \omega t dt$

But $\int_0^\infty e^{-at} \cos \omega t dt = \frac{a}{a^2+\omega^2}$

Differentiating both sides with respect to ω, we get

$$\int_0^\infty te^{-at} \sin \omega t dt = \frac{2a\omega}{(a^2+\omega^2)^2}$$

So $F_s(\omega) = \frac{2a\omega}{(a^2+\omega^2)^2}$

Example

Find the Fourier sine transform of $f(t) = \frac{1}{t}e^{-at}$ .Deduce that $F\{1/t\} = -i\pi$.

Solution

We know $\int_0^\infty e^{-at} \sin \omega t dt = \frac{\omega}{a^2+\omega^2}$

Integrating both sides with respect to a, we have

$$\int_0^\infty \frac{1}{t}e^{-at} \sin \omega t dt = \int_a^\infty \frac{\omega^2}{a^2+\omega^2} da = \frac{\pi}{2} - \tan^{-1}\frac{a}{\omega} = \tan^{-1}\frac{\omega}{a}$$

Thus $F_s(\omega) = \tan^{-1}\frac{\omega}{a}$

If $a \to 0$, then $F_s\left\{\frac{1}{t}\right\} = \frac{\pi}{2}$

which implies

$$F\left\{\frac{1}{t}\right\} = -2iF_s\left\{\frac{1}{t}\right\} = -i\pi$$



Example (AKTU 2018, 2020, 2021, 7 marks)

Find the Fourier cosine transform of $\frac{1}{1+x^2}$. *Hence, derive the Fourier sine transform of* $\frac{1}{1+x^2}$.

Solution

By definition, the Fourier cosine transform of f(x) is given by

$$F_c\{f(x)\} = F_c(\omega) = \int_0^\infty f(x)\cos\omega x dx = \int_0^\infty \frac{\cos\omega x}{1+x^2}dx \quad (1)$$

Differentiating both sides of (1) with respect to ω.

$$\frac{d}{d\omega}F_c(\omega) = \int_0^\infty \frac{1}{1+x^2}\frac{\partial}{\partial\omega}(\cos\omega x)dx = -\int_0^\infty \frac{x\sin\omega x}{1+x^2}dx = -\int_0^\infty \frac{x^2\sin\omega x}{x(1+x^2)}dx$$

$$= -\int_0^\infty \frac{[(1+x)^2-1]\sin\omega x dx}{x(1+x^2)} = -\int_0^\infty \frac{1}{x}\sin\omega x dx + \int_0^\infty \frac{\sin\omega x}{x(1+x^2)}dx$$

$$= -\frac{\pi}{2} + \int_0^\infty \frac{\sin\omega x}{x(1+x^2)}dx \quad (2)$$

Differentiating both sides of (2) with respect to ω again

$$\frac{d^2}{d\omega^2}F_c(\omega) = \int_0^\infty \frac{1}{x(1+x^2)}.\frac{\partial}{\partial\omega}(\sin\omega x)dx = \int_0^\infty \frac{\cos\omega x}{1+x^2}dx = F_c(\omega)\, by\,(1)$$

$$\therefore \quad (D^2-1)F_c(\omega) = 0 \quad (3)$$

The general solution of (3) is

$$F_c(\omega) = c_1e^{\omega} + c_2e^{-\omega} \quad (4)$$

$$\frac{d}{ds}[F_c(\omega)] = c_1e^{\omega} - c_2e^{-\omega} \quad (5)$$

When ω = 0, (1) and (4) give

$$F_c(0) = c_1 + c_2 = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} \quad (6)$$

Also $$F_c(0) = c_1 - c_2 = -\frac{\pi}{2} \text{ by (5)} \quad (7)$$

Solving (6) and (7), we get



$$c_1 = 0, c_2 = \pi/2 \tag{8}$$

From (1) and (4), we have

$$F_c\{f(x)\} = \frac{\pi}{2} e^{-\omega} \tag{9}$$

Now $$F_s\{\phi(x)\} = F_s\left\{\frac{x}{1+x^2}\right\} = \int_0^\infty \frac{x}{1+x^2}\sin\omega x dx = \frac{d}{d\omega}F_c(\omega) = -\frac{\pi}{2}e^{-\omega}$$

Example

Solve the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, t > 0$$

with u(x, 0) = f(x) and bounded (x, t), where

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

Solution

The solution of the given equation is

$$u(x,t) = \int_{-\infty}^{\infty} F(\omega) e^{-k\omega^2 t} e^{i\omega x} d\omega$$

where $$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{2\pi}\int_{-a}^{a} e^{-i\omega x} dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-a}^{a} = \frac{1}{\pi}.\frac{\sin a\omega}{\omega}$$

Thus $$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin a\omega}{\omega} e^{-k\omega^2 t} e^{i\omega x} d\omega$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin a\omega}{\omega} e^{-k\omega^2 t}(\cos\omega x + i\sin\omega x) d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\sin a\omega}{\omega} e^{-k\omega^2 t}\cos\omega x d\omega = \frac{1}{\pi}\int_0^{\infty} e^{-k\omega^2 t}.\frac{\sin(a+x)\omega + \sin(a-x)\omega}{\omega} d\omega$$

Example

Solve the heat equation



$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, x > 0$$

subject to the conditions

$$u(x,0) = e^{-x}$$
$$u(0,t) = 0, u(x,t) = 0, t \geq 0$$

Solution

This is a case of semi-infinite bar whose ends are kept at zero temperature. Since the boundary conditions do not involve derivative, we use sine transform. Taking Fourier sine transform, we get

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} \sin \omega x dx = k\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} \sin \omega x dx$$

or $$\frac{\partial}{\partial t} F_s(\omega) = k(i\omega)^2 F_s(\omega)$$

or $$\frac{\partial}{\partial t} F_s(\omega) + \omega^2 k F_s(\omega) = 0$$

The solution of this equation is

$$F_s(\omega) = Ae^{-e\omega^2 t} \qquad (1)$$

Taking Fourier sine transform of $u(x,0) = e^{-x}$,

We have $$F_s\{e^{-x}\} = \frac{\omega}{1+\omega^2}$$

Therefore, taking t = 0 in (1), we have

$$A = \frac{\omega}{1+\omega^2}$$

$\therefore$ $$F_s(\omega) = \frac{\omega}{1+\omega^2} e^{-k\omega^2 t}$$

Therefore, Fourier inversion formula for sine transform yields

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty} \frac{\omega}{1+\omega^2} e^{-k\omega^2 t} d\omega$$

Example

Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}; t > 0, 0 \le x \le \pi$

Subject to condition

$$u(x,0) = 2x, u(0,t) = u(\pi,t) = 0$$

Solution

The solution of given heat equation is

$$u(x,t) = \frac{2}{\pi}\int_0^\infty F_s(\omega)e^{-\omega^2 kt}\sin\omega d\omega$$

But ingretaion by parts yield

$$F_s(\omega) = \int_0^\infty 2x\sin\omega x dx = -\frac{2\pi}{\omega}\cos\omega\pi$$

Therefore $u(x,t) = -4\int_0^\infty \frac{\cos\omega\pi}{\omega}e^{-\omega^2 kt}\sin\omega x d\omega$



ASSIGNMENT

Q.1. (PTU 2018, 2 marks): If it is required to find the Fourier series of an odd function in (-π, π) then which formulae you will use?

Q.2. (PTU 2020, 2 marks): Find finite Fourier sine transform of f(t) = 1.

Q.3. (PTU 2018, 2 marks): What are Dirichlet's conditions for the expansion of f(x) as a Fourier series in (-π, π)?

Q.4. (PTU 2020, 2 marks): Define Euler formulae.

Q.5. (AKTU 2018, 2020, 2021, 7 marks): Find Fourier cosine transform of $\frac{1}{1+x^2}$ and hence find Fourier sine transform of $\frac{x}{1+x^2}$.

Answer: Solved in this module.

Q.6. (GTU 2019, 7 marks): Obtain Fourier series of the function $f(x)=\begin{cases}0 & -2\le x<0\\ 1 & 0\le x\le 2\end{cases}$.

Q.7. (GTU 2021, 7 marks): Expand $f(x)=\frac{1}{2}(\pi-x)$ as a Fourier series in the interval (0, 2π).

Q.8. (GTU 2014, 2019, 7 marks): Obtain Fourier series of the function f(x) = x + |x|, -π < x < π.

Answer: Solved in this module.

Q.9. (GTU 2020, 4 marks): Find Fourier series for $f(x)=x^2;-\pi\le x\le\pi$

Answer: Solved in this module.

Q.10. (GTU 2020, 7 marks): Find Fourier series in the interval (0, 2π) if

$$f(x)=\begin{cases}-\pi & 0<x<\pi\\ x-\pi & \pi<x<2\pi\end{cases}$$

and hence show that $\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$

Q.11. (GTU 2020, 7 marks): Find Fourier series for $f(x)=3x\left(\pi^2-x^2\right);-\pi<x<\pi$

Q.12. (GTU 2021, 7 marks): Find the Fourier series of

$$f(x)=\begin{cases}0 & -\pi<x<0\\ x & 0<x<\pi\end{cases}$$

Q.13. (GTU 2022, 7 marks): Find the Fourier series expansion of the periodic function f(x) = x - x^2 in the interval -π ≤ x ≤ π.

Answer:



$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\right]$$
$$+\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right]$$

Q.14. (GTU 2022, 7 marks): Find the Fourier series of the periodic function with a period 2 of

$$f(x) = \begin{cases} \pi & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$$

Q.15. (PTU 2019, 5 marks): Expand f(x) = xsinx; 0 < x < 2π as a Fourier series.

Answer: Solved in this module.

Q.16. (GTU 2014, PTU 2018, 5 marks): Find the Fourier series to represent $f(x) = \frac{1}{4}(\pi - x)^2; 0 \le x \le 2\pi$

Answer: Solved in this module.

Q.17. (GTU 2022, 7 marks): Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 - x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

Answer: Solved in this module.

Q.18. (GTU 2013, 2014, AKTU 2018, 2020, 7 marks): Express the function $f(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| < 1 \end{cases}$ as Fourier integral.

Hence evaluate $\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda$

Answer: Solved in this module.

Q.19. (AKTU 2021, 7 marks): If $F_c(p) = \frac{1}{2}\tan^{-1}\left(\frac{2}{p^2}\right)$ then find f(x).