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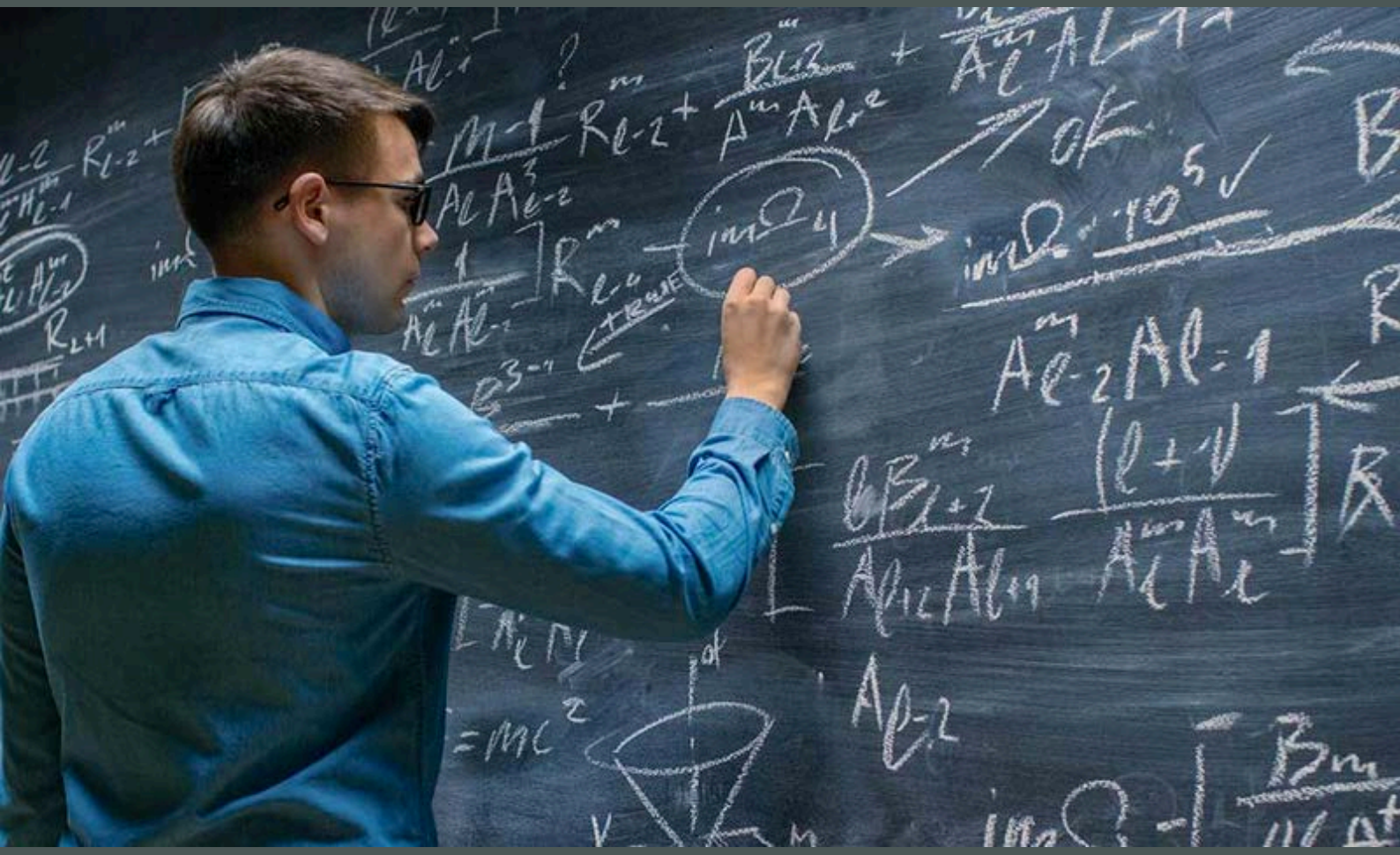

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COMPLEX ANALYSIS - II

TAYLOR'S THEOREM

Let f(z) be the analytic inside and on a simple closed curve C. Let a and a + h be two points inside C. Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^n}{n!}f^{(n)}(a) + ... \qquad (1)$$

or writing z = a + h, h = z – a,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + ... \frac{f^{(n)}(a)}{n!}(z-a)^n + ... \qquad (2)$$

This is called *Taylor's theorem* and the series (1) or (2) is called a *Taylor's series*.

The region of convergence of the series (2) is given by |z – a| < R, where the radius of convergence R is the distance from a to the nearest singularity of the function f(z). On |z – a| = R, the series may or may not converge. For |z – a| > R, the series diverges.

If the nearest singularity of f(z) is at infinity, the radius of convergence is infinite, i.e. the series converges for all z.

Example

Expand f(z) = sinz in a Taylor series about z = π/4.

Solution

f(z) = sinz, f'(z) = cosz, f''(z) = -sinz, f'''(z) = -cosz, $f^{iv}(z)$=sinz ……..

f(π/4) = √2/2, f'(π/4) = √2/2, f''(π/4) = -√2/2, f'''(π/4) = -√2/2, f^{iv}(π/4) = √2/2

Then since a = π/4

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)(z-a)^2}{2!} + \frac{f'''(a)(z-a)^3}{3!} + ...$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(z-\pi/4) - \frac{\sqrt{2}}{2.2!}(z-\pi/4)^2 - \frac{\sqrt{2}}{2.3!}(z-\pi/4)^3 + ...$$

$$= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4 - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + ...\right\}$$



Example

Let f(z) = ln(1+z), where we consider that branch which has the value zero when z = 0.

(a) Expand f(z) in a Taylor series about z = 0.

(b) Expand $\ln\left(\frac{1+z}{1-z}\right)$ *in a Taylor series about z = 0.*

Solution

(a) $f(z)=\ln(1+z)\Rightarrow f(0)=0$

$$f'(z)=\frac{1}{1+z}=(1+z)^{-1}\Rightarrow f'(0)=1$$

$$f''(z)=-(1+z)^{-2}\Rightarrow f''(0)=-1$$

$$f'''(z)=(-1)(-2)(1+z)^{-3}\Rightarrow f'''(0)=2!$$

....

....

$$f^{(n+1)}(z)=(-1)^n n!(1+z)^{-(n+1)}\Rightarrow f^{(n+1)}(0)=(-1)^n n!$$

Then $$f(z)=\ln(1+z)=f(0)+f'(0)z+\frac{f''(0)}{2!}z^2+\frac{f'''(0)}{3!}z^3+.....$$

$$=0+1.z+\frac{(-1)z^2}{2!}+\frac{2.z^3}{3!}+....=z-\frac{z^2}{2}-\frac{z^3}{3}+.....$$

(b) From above result, we have on replacing z by -z

$$\ln(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+.....$$

$$\ln(1-z)=-z-\frac{z^2}{2}-\frac{z^3}{3}-\frac{z^4}{4}-.....$$

$$\therefore\quad \ln\left(\frac{1+z}{1-z}\right)=2\left(z+\frac{z^3}{3}+\frac{z^5}{5}+.....\right)$$

Problem

Expand (z - 1)/(z + 10 in a Taylor series about the point z = 1.

Answer: $\frac{z-1}{z+1}=\frac{1}{2}(z-1)-\frac{1}{4}(z-1)^2+..$



Example

Expand $f(z)=\dfrac{z+1}{(z-3)(z-4)}$ *as a Taylor's series about z = 2.*

Solution

$$f(z)=\frac{z+1}{(z-3)(z-4)}=\frac{A}{z-3}+\frac{B}{z-4}$$

$\therefore$ $z + 1 = A(z - 4) + B(z - 3)$

Putting z = 3, A = -4

Putting z = 4, B = 5

$\therefore$ $$f(z)=\frac{4}{z-3}+\frac{5}{z-4}$$

Let z - 2 = t, then t = z + 2

$$f(z)=-\frac{4}{z-2-1}+\frac{5}{z-2-2}=-\frac{4}{t-1}+\frac{5}{t-2}=\frac{4}{1-t}-\frac{5}{2\left(1-\frac{t}{2}\right)}$$

$$=4(1-t)^{-1}-\frac{5}{2}\left(1-\frac{t}{2}\right)^{-1}=4\left(1-t+t^2-....\right)-\frac{5}{2}\left[1-\frac{t}{2}+\left(\frac{t}{2}\right)^2-...\right]$$

$$=\frac{3}{2}-\frac{11}{4}t+\frac{27}{8}t^2+....=\frac{3}{2}-\frac{11}{4}(z-2)+\frac{27}{8}(z-2)^2+...$$

Problem

Expand $f(z)=\dfrac{1}{z^2-4z+3}$ *about z = 4.*

Answer: $\frac{1}{3}-\frac{4}{9}(z-4)+\frac{13}{27}(z-4)^2-...$

Example

Find the Taylor's series to represent the function $\dfrac{z^2-1}{(z+2)(z+3)}$ *in* $|z| < 2$.



Solution

$$f(z)=\frac{z^2-1}{(z+2)(z+3)}=\frac{z^2-1}{z^2+5z+6}$$

Since the degrees of the numerator and denominator are same, partial fraction cannot be applied. Dividing to reduce the degree of numerator,

$$f(z)=1+\frac{-5z-7}{(z+2)(z+3)}$$

Let $$\frac{-5z-7}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$$

$$-5z - 7 = A(z + 3) + B(z + 2)$$

Putting z = -2, A = 3

Putting z = -3, B = -8

$$\therefore \quad f(z)=1+\frac{3}{z+2}-\frac{8}{z+3}$$

f (z) is not analytic at z = –2 and z = –3. But f (z) is analytic inside the region |z| < 2. Hence, f(z) in another way to make it analytic.

|z| < 2; |z/2| < 1; |z/3| < 1

Write f(z) as below

$$f(z)=1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

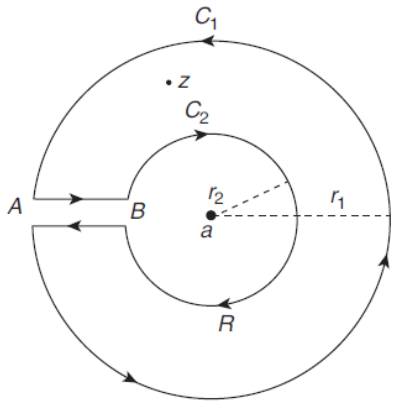
$$=1+\frac{3}{2}\left[1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\ldots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\ldots.\right]$$

$$=-\frac{1}{6}+\frac{5}{36}z+\frac{17}{216}z^2+\ldots$$

LAURENT'S SERIES

If f(z) is analytic on two concentric circles C_1 and C_2 with the centre at z = a and radii r_1, $r_2(r_2 < r_1)$ and in the annular region R between C_1 and C_2 then for all z in R,





$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + ...$$

Example (GTU 2013)

Find Laurent's series expansion in powers of z that represent $f(z) = \frac{1}{z^2(1-z)}$ *for the following domains: (i) |z| < 1 (ii) |z| > 1.*

Solution

$$f(z) = \frac{1}{z^2(1-z)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{1-z}$$

$$1 = Az(1 - z) + B(1 - z) + Cz^2$$

Putting z = 0, B = 1

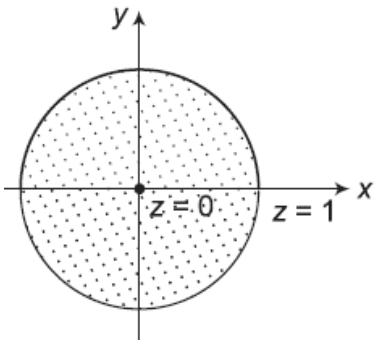
Putting z = 1, C = 1

Putting z = -1, 1 = -2A + 2B + C giving A = 1.

$$\therefore \qquad f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{1-z}$$

(i) |z| < 1

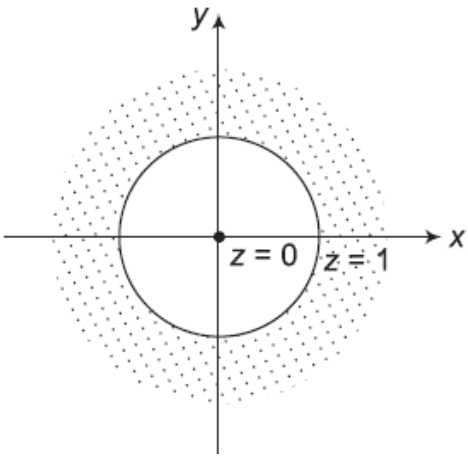
See following figure.



f(z) is analytic in the region $|z| < 1$ about $z = 0$.

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + (1-z)^{-1} = \frac{1}{z} + \frac{1}{z^2} + (1 + z + z^2 + \ldots)$$

(ii) $|z| > 1 \Rightarrow |1/z| < 1$



$$f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{-z\left(-\frac{1}{z} + 1\right)}$$

$$= \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} = \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right)$$

$$= \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \ldots = -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \ldots$$

Example (AKTU 2019)

Find the Taylor and Laurent series which represent the function $\frac{1}{z^2 - 3z + 2}$ *in the region (a)* $|z| < 1$ *(b)* $1 < |z| < 2$ *(c)* $|z| > 2$.

Solution

$$f(z)=\frac{1}{z^2-3z+2}=\frac{1}{(z-2)(z-1)}=\frac{1}{z-2}-\frac{1}{z-1}$$

(a) $|z| < 1 \Rightarrow |z/2| < 1$

$$f(z)=\frac{1}{1-z}-\frac{1}{2\left(1-\frac{z}{2}\right)}=(1-z)^{-1}-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

Now expand. This is Taylor's series.

(b) For $1 < |z| < 2 \Rightarrow |z|/2 < 1$

$$f(z)=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{2}\left(1-\frac{1}{z}\right)^{-1}$$

This is Laurent series. Now you can expand it as before.

(c) For $|z| > 2 \Rightarrow 2/|z| < 1, 1/|z| < 1/2 < 1$

$$f(z)=\frac{1}{2}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

This is Laurent series. Now you can expand it as before.

Example

Find Laurent series about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series.

(i) $\frac{e^{2z}}{(z-1)^3}; z=1$

(ii) $(z-3)\sin\frac{1}{z+2}; z=-2$

(iii) $\frac{z-\sin z}{z^3}; z=0$

(iv) $\frac{z}{(z+1)(z+2)}; z=-2$

(v) $\frac{1}{z^2(z-3)^2}; z=3$



Solution

(i) $\dfrac{e^{2z}}{(z-1)^3}; z=1$

Let $z - 1 = u$

Then $z = 1 + u$ and

$$\frac{e^{2z}}{(z-1)^3}=\frac{e^{2+2u}}{u^3}=\frac{e^2}{u^3}e^{2u}=\frac{e^2}{u^3}\left\{1+2u+\frac{(2u)^2}{2!}+\frac{(2u)^3}{3!}+\frac{(2u)^4}{4!}+.....\right\}$$

$$=\frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2e^2}{z-1}+\frac{4e^2}{3}+\frac{2e^2}{3}(z-1)+.....$$

$z = 1$ is a pole of order 3.

The series converges for all values of $z \neq 1$.

(ii) $(z-3)\sin\dfrac{1}{z+2}; z=-2$

Let $z + 2 = u$ or $z = u - 2$

Then $$(z-3)\sin\frac{1}{z+2}=(u-5)\sin\frac{1}{u}=(u-5)\left\{\frac{1}{u}-\frac{1}{3!u^3}+\frac{1}{5!u^5}-....\right\}$$

$$=1-\frac{5}{u}-\frac{1}{3!u^2}+\frac{5}{3!u^3}+\frac{1}{5!u^4}-.....$$

$$=1-\frac{5}{z+2}-\frac{1}{6(z+2)^2}+\frac{5}{6(z+2)^3}+\frac{1}{120(z+2)^4}-.....$$

$z = -2$ is an essential singularity.

The series converges for all values of $z \neq -2$.

(iii) $\dfrac{(z-\sin z)}{z^3}; z=0$

$$\frac{z-\sin z}{z^3}=\frac{1}{z^3}\left\{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+.....\right)\right\}$$

$$=\frac{1}{z^3}\left\{\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}-.....\right\}=\frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-.....$$

$z = 0$ is a removable singularity.

The series converges for all values of z.



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(iv) $\dfrac{z}{(z+1)(z+2)}; z=-2$

Let z + 2 = u.

Then $\dfrac{z}{(z+1)(z+2)} = \dfrac{u-2}{(u-1)u} = \dfrac{2-u}{u}.\dfrac{1}{1-u} = \dfrac{2-u}{u}(1+u+u^2+u^3+...)$

$$= \frac{2}{u}+1+u+u^2+.... = \frac{2}{z+2}+1+(z+2)+(z+2)^2+.....$$

z = -2 is a pole of order 1, or simple pole.

The series converges for all values of z such that $0 < |z + 2| < 1$.

(v) $\dfrac{1}{z^2(z-3)^2}; z=3$

Let z – 3 = u.

Then by binomial theorem

$$\frac{1}{z^2(z-3)^2} = \frac{1}{u^2(3+u)^2} = \frac{1}{9u^2(1+u/3)^2}$$

$$= \frac{1}{9u^2}\left\{1+(-2)\left(\frac{u}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3\right\}+...$$

$$= \frac{1}{9u^2}-\frac{2}{27u}+\frac{1}{27}-\frac{4}{243}u+....$$

$$= \frac{1}{9(z-3)^2}-\frac{2}{27(z-3)}+\frac{1}{27}-\frac{4(z-3)}{243}+.....$$

z = 3 is a pole of order 2.

The series converges for all values of z such that $0 < |z - 3| < 3$.

Example

Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ *in a Laurent series valid for (a) 1 < z < 3 (b) |z| > a (c) 0 < |z + 1| < 2 (d) |z| < 1*

Solution

(a) Resolving into partial fraction



$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

If $|z| > 1$

$$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + ...\right) = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + ...$$

If $|z| < 3$

$$\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + ...\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + ...$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| < 3$ i.e. $1 < |z| < 3$, is

$$.....- \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - .....$$

(b) If $|z| > 1$, we have as in above part

$$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + ...\right) = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + ...$$

If $|z| > 3$

$$\frac{1}{2(z+3)} = \frac{1}{2z(1+3/z)} = \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + ...\right) = \frac{1}{2z} - \frac{3}{2z^2} + \frac{9}{2z^3} - \frac{27}{2z^4} + ...$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| > 3$ i.e. $|z| > 3$ is by subtraction

$$\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + ...$$

(c) Let $z + 1 = u$. Then

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + ...\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + ...$$

valid for $|u| < 2,\ u \neq 0$ or $0 < |z + 1| < 2$.

(d) If $|z| < 1$

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \frac{1}{2}(1 - z + z^2 - z^3 + ...) = \frac{1}{2} - \frac{1}{2}z + \frac{1}{2}z^2 - \frac{1}{2}z^3 + ...$$



If |z| < 3, we have by part (a)

$$\frac{1}{2(z+3)} = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \ldots$$

Then the required Laurent expansion, valid for both |z| < 1 and |z| < 3, i.e. |z| < 1, is by subtraction

$$\frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \ldots..$$

Example (UTU 2013, 2014, 5 marks)

Find the Laurent's expansion for $f(z) = \frac{7z-2}{z^3 - z^2 - 2z}$ *in the region (i) 0 < |z +1|<1 (ii) 1 < |z + 1| < 3.*

Solution

$$f(z) = \frac{7z-2}{z^3 - z^2 - 2z} = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2} = \frac{1}{(z+1)-1} - \frac{3}{z+1} + \frac{2}{(z+1)-3}$$

(i) $0 < |z+1| < 1$

$$f(z) = -\{1-(z+1)\}^{-1} - \frac{3}{z+1} - \frac{2}{3}\left\{1 - \left(\frac{z+1}{3}\right)\right\}^{-1}$$

$$= -\left[1 + (z-1) + (z-1)^2 = (z-1)^3 + \ldots\right] - \frac{3}{z+1} - \frac{2}{3}\left[1 + \frac{(z+1)}{3} + \frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + ..\right]$$

(ii) $1 < |z+1| < 3$.

$$f(z) = \frac{1}{(z+1)-1} - \frac{3}{z+1} + \frac{2}{(z+1)-3}$$

Taking common, bigger term out of |z + 1| and 1 in first fraction, here |z + 1| is bigger than 1, so we take |z + 1| common from the first fraction.

Similarly, we take 3 common from the third fraction as 3 is bigger than |z + 1|.

$\therefore$

$$f(z) = \frac{1}{z+1}\left[\frac{1}{1-\left(\frac{1}{z+1}\right)}\right] - \frac{3}{z+1} - \frac{2}{3}\left[\frac{1}{1-\left(\frac{z+1}{3}\right)}\right]$$



$$= \frac{1}{(z+1)}\left(1-\frac{1}{z+1}\right)^{-1} - \frac{3}{z+1} - \frac{2}{3}\left\{1-\left(\frac{z+1}{3}\right)\right\}^{-1}$$

$$= \frac{1}{z+1}\left[1+\frac{1}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\ldots\right]$$
$$-\frac{3}{z+1}-\frac{2}{3}\left[1+\frac{z+1}{3}+\frac{(z+1)^2}{9}+\frac{(z+1)^3}{27}+\ldots\right]$$

Problem

Expand f(z) = 1/(z – 3) in Laurent series valid for (a) |z| < 3 (b) |z| > 3

Answer: (a) $-\frac{1}{3}-\frac{1}{9}z-\frac{1}{27}z^2-\frac{1}{81}z^3-.....$

(b) $z^{-1}+3z^{-2}+9z^{-3}+27z^{-4}+.....$

SINGULAR POINTS

A point at which the function f (z) is not analytic is known as a singular point or singularity of the function.

Isolated singularity

A singular point z = a is known as an isolated singularity if there is no other singularity within a small circle surrounding the point z = a; otherwise it is called a non-isolated singularity.

$$f(z)=\frac{z^2}{z-2}$$

has isolated singularity at z = 2.

Removable singularity

An isolated singular point z = a is known as a removable singularity if $\lim_{z\to a} f(z)$ exists or there is no negative power term of (z – a) in Laurent's series of f (z), i.e., $b_n = 0$.

For example, $f(z)=\frac{\sin z}{z}$

z = 0 is a singularity of f(z).

$$\lim_{z\to 0}\frac{\sin z}{z}=1$$

∴ f(z) has a removable singularity at z = 0.



Essential singularity

A singular point z = a is known as an essential singularity if the number of negative power terms of (z – a) in Laurent's series is infinite.

For example $f(z) = e^{1/(z-2)} = 1 + \frac{1}{z-2} + \frac{1}{2!}\frac{1}{(z-2)^2} + \ldots$

Since the number of negative power terms of (z – 2) is infinite, z = 2 is an essential singularity.

RESIDUES

The coefficient of $\frac{1}{z-a}$ in the Laurent series expansion of f(z) is known as the residue of f(z) at z = a.

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots.. + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots$$

Residue (at z = a) = coefficient of 1/(z - a)

$$= \frac{1}{2\pi i}\oint_C f(z)dz$$

Calculation of Residues

In the case where z = a is a pole of order k there is a simple formula for a_{-1} given by

$$a_{-1} = \lim_{z\to a} \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\{(z-a)^k f(z)\}$$

If k = 1 (simple pole) the result is especially simple and is given by

$$a_{-1} = \lim_{z\to a}(z-a)f(z)$$

Example

Find residues of $f(z) = \frac{z}{(z-1)(z+1)^2}$ *at all poles.*

Solution

Here z = 1 and z = -1 are poles of orders one and two respectively. We have using (4) and (5) with k = 2,



Residue at z = 1 is $\lim_{z\to 1}(z-1)\left\{\frac{z}{(z-1)(z+2)^2}\right\}=\frac{1}{4}$

Residue at z = -1 is $\lim_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2\left(\frac{z}{(z-1)(z+1)^2}\right)\right\}=-\frac{1}{4}$

If z = a is an essential singularity, the residue can sometimes be found by using known series expansions.

Example (JNTUK 2021, 2 marks)

Find residue of f(z) = $e^{-1/z}$.

Solution

Here z = 0 is an essential singularity and from the known expansion for e^u with u = -1/z, we find

$$e^{-1/z}=1-\frac{1}{z}+\frac{1}{2!z^2}-\frac{1}{3!}z^3+.....$$

from which we see that the residue at z = 0 is the coefficient of 1/z and equals –1.

Example (UTU 2017, 5 marks)

Find the residues of $f(z)=\frac{z^2-2z}{(z+1)^2(z^2+4)}$ *at all its poles in the finite plane.*

Solution

Residue at z = -1 is

$$\lim_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2.\frac{z^2-2z}{(z+1)^2(z^2+4)}\right\}=\lim_{z\to -1}\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}=-\frac{14}{25}$$

Residue at z = 2i is

$$\lim_{z\to 2i}\left\{(z-2i).\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\}=\frac{-4-4i}{(2i+1)^2(4i)}=\frac{7+i}{25}$$

Residue at z = -2i is

$$\lim_{z\to -2i}\left\{(z+2i).\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\}=\frac{-4+4i}{(-2i+1)^2(-4i)}=\frac{7-i}{25}$$



Example (JNTUK 2020, 2 marks)

Determine the residue of f(z) = z.cos(1/z) at z = 0.

Solution

Expanding the function in powers of 1/z, we have

$$z\cos\frac{1}{z} = z\left[1-\frac{1}{2z^2}+\frac{1}{4!z^4}-\ldots\right] = z-\frac{1}{2z}+\frac{1}{24z^3}-\ldots$$

This is Laurent's expansion about z = 0.

The coefficient of 1/z in it is (-1/2). So, the residue of zcos(1/z) at z = 0 is (-1/2).

Example (UTU 2015, 5 marks)

Determine the poles and residue at each pole of the function

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$

Solution

The function f(z) has a pole of order 2 at z = 1 and a simple pole at z = -2.

Residue of f(z) at z = 1 is

$$= \lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)] = \lim_{z\to 1}\frac{d}{dz}\left(\frac{z^2}{z+2}\right) = \lim_{z\to 1}\frac{(z+2).2z-z^2.1}{(z+2)^2} = \frac{5}{9}$$

Residue of f(z) at z = -2 is

$$= \lim_{z\to -2}[(z+2)f(z)] = \lim_{z\to -2}\frac{z^2}{(z-1)^2} = \frac{4}{9}$$

Since both the poles lie inside the given curve C: |z| = 3.

$$\therefore \qquad \oint_C \frac{z^2}{(z-1)^2(z+2)}dz = 2\pi i\left[\frac{5}{9}+\frac{4}{9}\right] = 2\pi i$$

Problem

For each of the following functions, determine the poles and the residues at the poles

(a) $\dfrac{2z+1}{z^2-z-2}$



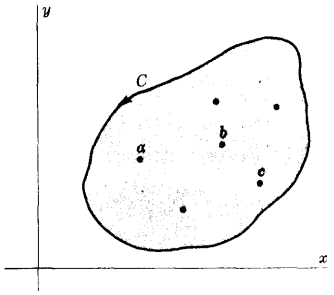
(b) $\left(\frac{z+1}{z-1}\right)^2$

(c) $\frac{\sin z}{z^2}$

Answer: (a) z = -1,2;1/3,5/3 (b) z = 1;4 (c) z = 0;1

Residue Theorem

Let f(z) be single values and analytic inside and on a simple closed curve C except at the singularities a, b, c,…. inside C which have residues given by a_{-1}, b_{-1}, c_{-1}….. (see figure). Then the residue theorem states that



$$\oint_C f(z)dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + ...)$$

i.e. the integral of f(z) around C is 2πi times the sum of the residues of f(z) at the singularities enclosed by C.

Example (AKTU 2018, 7 marks)

State and prove the Cauchy integral formula. Also evaluate $\int_C \frac{1}{(z^2+4)^2}dz = \frac{\pi}{16}$ *where C is the circle |z - i| = 2.*

Solution

$$f(z) = \frac{1}{(z^2+4)} = \frac{1}{(z+2i)^2(z-2i)^2} \qquad [i^2 = -1,\ \therefore\ 4 = 2^2 = -(2i)^2]$$

Thus we see that f(z) has two double poles at z = 2i and z = -2i, out of which z = 2i lies within |z - i| = 2.

$$\text{Res}(2i) = \lim_{z\to 2i}\frac{d}{dz}[(z-2i)^2.f(z)] = \lim_{z\to 2i}\frac{d}{dz}\left[\frac{1}{(z+2i)^2}\right] = \lim_{z\to 2i}\left[\frac{-2z=4i}{(z+2i)^4}\right] = \frac{-i}{32}$$

Hence, by Cauchy's residue theorem

$$\oint_C \frac{dz}{(z^2+4)^2} = 2\pi i\left(\frac{-i}{32}\right) = \frac{\pi}{16}$$

Example (AKTU 2019, 3.5 marks)

Find the residue $f(z)=\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ *at its pole and hence evaluate* $\int_C f(z)dz$ *where C is the circle |z| = 5/2.*

Solution

Poles of f(z) are given by $(z - 1)^4(z - 2)(z - 3) = 0$. These are 1, 2 and 3.

z = 1 is a pole of order 4 while z = 2 and z = 3 are simple poles.

Residue of f(z) at z = 2 is

$$R_1=\lim_{z\to 0}(z-2)\frac{z^3}{(z-1)^4(z-2)(z-3)}=\lim_{z\to 2}\frac{z^3}{(z-1)^4(z-3)}=\frac{8}{(-1)}=-8$$

Residue of f(z) at z = 3 is

$$R_2=\lim_{z\to 3}(z-3)\frac{z^3}{(z-1)^2(z-2)(z-3)}=\lim_{z\to 3}\frac{z^3}{(z-1)^4(z-2)}=\frac{27}{16}$$

Residue of f(z) at z = 1 is

$$R_3=\frac{1}{(4-1)!}\left[\frac{d^3}{dz^3}\left\{(z-1)^4\frac{z^3}{(z-1)^4(z-20(z-3)}\right\}\right]_{z=1}$$

$$=\frac{1}{6}\left[\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}\right]_{z=1}=\frac{1}{6}\left[\frac{d^3}{dz^3}\left\{z+5+\frac{19z-30}{z^2-5z+6}\right\}\right]_{z=1}$$

$$=\frac{1}{6}\left[\frac{d^3}{dz^3}\left\{z+5+\frac{19}{z-3}+8\left(\frac{1}{z-3}-\frac{1}{z-2}\right)\right\}\right]_{z=1}$$

$$=\frac{1}{6}\left[\frac{d^3}{dz^3}\left\{z+5+\frac{27}{z-3}-\frac{8}{z-2}\right\}\right]_{z=1}$$

$$=\frac{1}{6}\left[\frac{d^2}{dz^2}\left\{1-\frac{27}{(z-3)^2}+\frac{8}{(z-2)^2}\right\}\right]_{z=1}$$

$$=\frac{1}{6}\left[\frac{d}{dz}\left\{\frac{54}{(z-3)^2}-\frac{16}{(z-2)^3}\right\}\right]_{z=1}$$

$$=\frac{1}{6}\left[\frac{-162}{(z-3)^4}+\frac{48}{(z-2)^4}\right]_{z=1}=\frac{101}{16}$$



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The given curve C = | z | = 5/2 is a circle with centre at (0, 0) and radius 5/2.

Clearly only the poles z = 1 and z = 2 lie inside this circle.

Hence By Cauchy's Residue theorem,

$$\oint_C f(z)dz = 2\pi i(R_3 + R_1) = 2\pi i\left(\frac{101}{16} - 8\right) = -\frac{27\pi i}{8}$$

Example

Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz$ *around the circle C with equation* $|z| = 3$.

Solution

The integrand $\frac{e^{zt}}{z^2(z^2+2z+2)}dz$ has a double pole at z = 0 and two simple poles at z = -1 ± i [roots of $z^2 + 2z + 2 = 0$]. All these poles are inside C.

Residue at z = 0 is

$$\lim_{z\to 0}\frac{d}{1!dz}\left\{z^2\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \lim_{z\to 0}\frac{(z^2+2z+2)-(e^{zt})(2z+2)}{(z^2+2z+2)^2} = \frac{t-1}{2}$$

Residue at z = -1 + i is

$$\lim_{z\to -1+i}\left\{[z-(-1+i)]\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \lim_{z\to -1+i}\left\{\frac{e^{zt}}{z^2}\right\}\lim_{z\to -1+i}\left\{\frac{z+1-i}{z^2+2z+2}\right\}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2}\cdot\frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

Residue at z = -1 – i is (in similar fashion)

$$\frac{e^{(-1-i)t}}{4}$$

Then by the residue theorem

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = 2\pi i(\text{sum of residues})$$



$$= 2\pi i\left\{\frac{t-1}{2}+\frac{e^{(-1+i)t}}{4}+\frac{e^{(-1-i)t}}{4}\right\} = 2\pi i\left\{\frac{t-1}{2}+\frac{1}{2}e^{-t}\cos t\right\}$$

i.e. $$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = \frac{t-1}{2}+\frac{1}{2}e^{-t}\cos t$$

Problem

Evaluate $\oint_C \frac{e^z dz}{\cosh z}$ *around the circle C defined by* $|z| = 5$.

Answer: $8\pi i$

Problem

Evaluate $\oint_C e^{-1/z}\sin(1/z)dz$ *where C is the circle* $|z| = 1$.

Answer: $2\pi i$

DEFINITE INTEGRALS OF CLOSED CURVES (CONTOUR INTEGRALS)

The evaluation of definite integrals is often achieved by using the residue theorem together with a suitable function f(z) and a suitable closed path or *contour* C.

Example

Evaluate $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$

Solution

Let $z = e^{i\theta}$.

Then $$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

and $$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

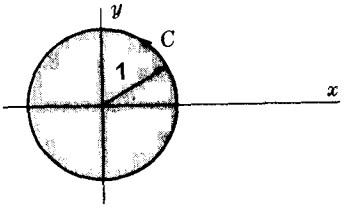
$$dz = iz d\theta$$

So that

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_C \frac{dz/iz}{3-2(z+z^{-1})/2+(z-z^{-1})/2i} = \oint_C \frac{2dz}{(1-2i)z^2+6iz-1-2i}$$



where C is the circle of unit radius with centre at the origin.



The poles of $\frac{2}{(1-2i)z^2+6iz-1-2i}$ are the simple poles

$$z=\frac{-6i\pm\sqrt{(6i)^2-4(1-2i)(-1-2i)}}{2(1-2i)}=\frac{-6i\pm 4i}{2(1-2i)}=2-i,(2-i)/5$$

☞ *Here we used the formula, $x = (-b \pm \sqrt{(b^2 - 4ac)})/2a$. to calculate the roots for a quadratic equation $ax^2 + bx + c = 0$.*

Only (2 – i)/5 lies inside C.

Residue at (2 – i)/5

$$=\lim_{z\to(2-i)/5}\{z-(2-i)/5\}\left\{\frac{2}{(1-2i)z^2+6iz-1-2i}\right\}=\lim_{z\to(2-i)/5}\frac{2}{2(1-2i)z+6i}=\frac{1}{2i}$$

Then $$\oint_C\frac{2dz}{(1-2i)z^2+6iz-1-2i}=2\pi i\left(\frac{1}{2i}\right)=\pi$$

Problem

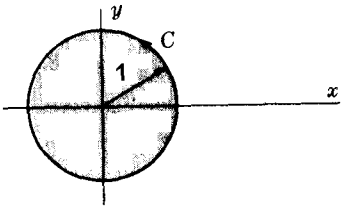
Show that $\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta=\frac{\pi}{12}$

Example

Evaluate $\int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^2+2x+5}dx$.

Solution

Consider $\oint_C\frac{ze^{i\pi z}}{z^2+2z+5}dz$ where C is the contour of given figure.



The integrand has simple poles at z = -1 ± 2i, but only z = -1 ± 2i lies inside C.

Residue at z = -1 ± 2i is

$$\lim_{z\to -1+2i}\left\{(z+1-2i).\frac{ze^{i\pi z}}{z^2+2z+5}\right\}=(-1+2i)\frac{e^{-i\pi-2\pi}}{4i}$$

Then $$\oint_C \frac{ze^{i\pi z}}{z^2+2z+5}dz=2\pi i(-1+2i)\left(\frac{e^{-i\pi-2\pi}}{4\pi}\right)=\frac{\pi}{2}(1-2i)e^{-2\pi}$$

or $$\int_{-R}^{R}\frac{xe^{i\pi x}}{x^2+2x+5}dx+\int_{\Gamma}\frac{ze^{i\pi z}}{z^2+2z+5}dz=\frac{\pi}{2}(1-2i)e^{-2\pi}$$

i.e. $$\int_{-R}^{R}\frac{x\cos\pi x}{x^2+2x+5}dx+i\int_{-R}^{R}\frac{x\sin\pi x}{x^2+2x+5}+\int_{\Gamma}\frac{ze^{i\pi z}}{z^2+2z+5}=\frac{\pi}{2}(1-2i)e^{-2\pi}$$

Taking the limit as $R\to\infty$ and using relation $\lim_{R\to\infty}\int_{\Gamma}e^{imz}F(z)dz=0$ to show that the integral around Γ approaches zero, this becomes

$$\int_{-\infty}^{\infty}\frac{x\cos\pi x}{x^2+2x+5}dx+i\int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^2+2x+5}dx=\frac{\pi}{2}e^{-2\pi}-i\pi e^{-2\pi}$$

Equating real and imaginary parts

$$\int_{-\infty}^{\infty}\frac{x\cos\pi x}{x^2+2x+5}=\frac{\pi}{2}e^{-2\pi};\int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^2+2x+5}dx=-\pi e^{-2\pi}$$

Example

Prove that $\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta=\frac{2\pi}{b^2}\{a-\sqrt{a^2-b^2}\}$ *where* $a > b > 0$.

Solution

We have $$I=\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta$$

$$= \int_C \frac{\left\{\frac{1}{2i}\left(z-\frac{1}{z}\right)\right\}^2}{a+\frac{1}{2}b\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{iz} = -\frac{1}{2i}\int_C \frac{(z^2-1)^2}{z^2(bz^2+2az+b)}dz$$

$$= -\frac{1}{2ib}\int_C \frac{(z^2-1)^2}{z^2[z^2+(2a/b)z+1]}dz = -\frac{1}{2ib}\int_C f(z)dz \text{ say}$$

where C is the unit circle.

f(z) has a pole of order two at z = 0 and simple poles at z = α, β, where α, β are the roots of $z^2+(2a/b)z+1=0$

or $$z = \frac{-a+\sqrt{(a^2-b^2)}}{b} = \alpha, z = \frac{-a-\sqrt{(a^2-b^2)}}{b} = \beta$$

Since a > b > 0, |β| > 1. Also |αβ| = 1 therefore |α| < 1.

Thus z = α is a simple pole and z = 0 are double poles inside C.

Residue at z = α is $$\lim_{z\to\alpha}(z-\alpha)\left[-\frac{1}{2ib}\cdot\frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}\right]$$

$$= -\frac{(\alpha^2-1)^2}{2ib\alpha^2(\alpha-\beta)} = -\frac{(\alpha-1/\alpha)^2}{2ib(\alpha-\beta)} = -\frac{(\alpha-\beta)^2}{2ib(\alpha-\beta)} \qquad [\alpha = 1/\beta]$$

$$= -\frac{1}{2ib}(\alpha-\beta) = -\frac{1}{2ib}\cdot\frac{2}{b}\sqrt{(a^2-b^2)} = \frac{i}{b^2}\sqrt{(a^2-b^2)}$$

and residue at z = 0 is coeff of 1/z in $-\frac{(z^2-1)^2}{2ibz^2\{z^2+(2a/b)z+1\}}$

$$= \text{coeff of } 1/z \text{ in } -\frac{1}{2ibz^2}(1-2z^2+z^4)\left(1+\frac{2a}{b}z+z^2\right)^{-1}$$

$$= \text{coeff of } 1/z \text{ in } -\frac{1}{2ibz^2}(1-2z^2+z^4)\left(1-\frac{2a}{b}z-....\right)$$

$$= \frac{a}{ib^2} = -\frac{ai}{b^2}$$

$\therefore$ $$\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta = 2\pi i \text{ . sum of residues at poles inside C}$$



$$= 2\pi i\left[\frac{1}{b^2}\sqrt{(a^2-b^2)}-\frac{ai}{b^2}\right]=\frac{2\pi}{b^2}[a-\sqrt{(a^2-b^2)}]$$

Example (UTU 2017, AKTU 2019, 7 marks)

Using complex integration method, evaluate $\int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}$.

Solution

Let $$I=\int_0^{2\pi}\frac{\cos 2\theta d\theta}{5+4\cos\theta}=\text{real part of }\int_0^{2\pi}\frac{e^{2i\theta}d\theta}{5+4\cos\theta}$$

$$= RP\int_0^{2\pi}\frac{e^{2i\theta}d\theta}{5+4\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}=RP\oint_C\frac{z^2}{5+2\left(z+\frac{1}{z}\right)}.\frac{dz}{iz}$$

Writing $z = e^{i\theta}$, $d\theta = dz/iz$

where C is the circle |z| = 1 = real part (RP) of $\oint_C f(z)dz$

where $$f(z)=\frac{z^2}{2i\left(z^2+\frac{5}{2}z+1\right)}$$

By Cauchy's residue theorem

$$\oint_C f(z)dz = 2\pi i(\text{Sum of residues}) \qquad (1)$$

Now, the poles of f(z) are given by

$$z^2+\frac{5}{2}z+1=0$$

Roots of this quadratic equation are z = -1/2 and z = -2. These are simple poles of f(z).

But pole z = 1/2 lies within C.

Residue of f(z) at the simple pole z = -1/2

$$= \lim_{z\to(-1/2)}\left(z+\frac{1}{2}\right)f(z)=\lim_{z\to(-1/2)}\left(z+\frac{1}{2}\right).\frac{z^2}{2i(z+2)\left(z+\frac{1}{2}\right)}$$



$$= \frac{(1/4)}{2i\left(-\frac{1}{2}+2\right)} = \frac{1}{12i} = \text{sum of residues}$$

$\therefore$ From (1) $\oint_C f(z)dz = 2\pi i\left(\frac{1}{12i}\right) = \frac{\pi}{6}$

Problem

Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}$

Answer: π/12

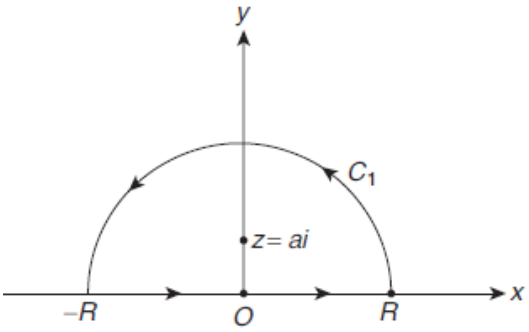
Example (UTU 2014, 10 marks)

Evaluate $\int_0^{\infty} \frac{\cos mx}{x^2+a^2}dx$, *a > 0, m > 0 using contour integration.*

Solution

Let $\oint_C f(z)dz = \oint \frac{e^{imz}}{z^2+a^2}dz$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from -R to R.



Let $f(z) = \frac{e^{imz}}{z^2+a^2}$; poles are z = ± ai

$\therefore$ $f(z) = \frac{e^{imz}}{(z+ai)(z-ai)}$

The pole z = ai lies inside C.

Residue at z = ai

$$= \lim_{z \to ai}(z-ai)\,f(z) = \lim_{z \to ai}\frac{e^{imz}}{(z+ai)} = \frac{e^{im(ai)}}{(ai+ai)}$$

Now $\int_{-\infty}^{\infty} f(x)dx = 2\pi i(\text{sum of residues})$

$\therefore$ $$\int_{-\infty}^{\infty}\frac{e^{imx}}{x^2+a^2}dx = 2\pi i\left(\frac{e^{-ma}}{2ai}\right) = \frac{\pi e^{-ma}}{a}$$

$$\int_{-\infty}^{\infty}\frac{(\cos mx + i\sin mx)}{x^2+a^2}dx = \frac{\pi e^{-ma}}{a}$$

Comparing real parts

$$\int_{-\infty}^{\infty}\frac{\cos mx}{x^2+a^2}dx = \frac{\pi e^{-ma}}{a}$$

$$2\int_{0}^{\infty}\frac{\cos mx}{x^2+a^2}dx = \frac{\pi e^{-ma}}{a} \qquad \left[\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \text{ if } f(-x) = f(x)\right]$$

$\therefore$ $$\int_{0}^{\infty}\frac{\cos mx}{x^2+a^2}dx = \frac{\pi e^{-ma}}{2a}$$

Problem

Show that $\int_0^{\infty}\frac{\cos mx}{x^2+1}dx = \frac{\pi}{2}e^{-m}$ *, m > 0.*

Example

Apply the calculus of residue to evaluate

$$\int_0^{2\pi}\frac{\cos n\theta}{1+2a\cos\theta+a^2}d\theta$$ *when n is a positive integer and* $a^2 < 1$.

Solution

Let $$I = \int_0^{2\pi}\frac{e^{in\theta}}{1+2a\cos\theta+a^2}d\theta$$

Putting $z = e^{i\theta}$ so that $d\theta = dz/iz$ and $\cos\theta = \frac{z+z^{-1}}{2}$

We get $$I = \oint_C \frac{z^n}{[1+a(z+z^{-1})+a^2]}.\frac{dz}{iz}$$ where C is the circle |z| = 1.



$$= \frac{1}{i}\oint_C \frac{z^n}{(1+a^2)z+az^2+a}dz = \frac{1}{ai}\oint_C \frac{z^n}{z^2+az+(z/a)+1}dz$$

$$= \frac{1}{ai}\oint_C \frac{z^n}{(z+a)\left(z+\frac{1}{a}\right)}dz = \frac{1}{ai}\oint_C f(z)dz \text{ where } f(z) = \frac{z^n}{(z+a)\left(z+\frac{1}{a}\right)}$$

The poles of f(z) are given by $(z+a)\left(z+\frac{1}{a}\right)=0$

Hence, f(z) has simple poles at z = -a, (-1/a) of which z = -a lies inside C as $a^2 < 1$.

Now $$\underset{z\to -a}{\text{Res}}\, f(z) = \lim_{z\to -a}(z+a)\frac{z^n}{(z+a)\left(z+\frac{1}{a}\right)} = \lim_{z\to -a}\frac{z^n}{z+(1/a)} = \frac{(-a)^n}{-a+(1/a)} = \frac{(-1)^n a^{n+1}}{1-a^2}$$

Thus by cauchy's residue theorem, we have

$$\oint_C f(z)dz = 2\pi i \underset{z\to -a}{\text{Res}}\, f(z) = 2\pi i\left[\frac{(-1)^n a^{n+1}}{1-a^2}\right] = \frac{2\pi i(-1)^n a^{n+1}}{1-a^2}$$

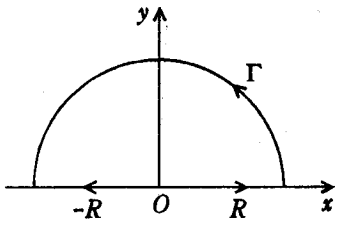
Example

If a > 0, prove that $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$

Solution

Consider $$\int_C f(z)dz = \int_C \frac{dz}{(a^2+z^2)}$$

where C is the contour consisting of a large semi circle Γ of radius R together with real axis from –R to R.



z = ai, -ai are the poles of f(z) of second order. Out of these only z = ai lies inside C.

Residue at z = ai is

$$\phi'(ai) = [(d/dz)(z+ai)^{-2}]_{z=ai}$$

$$= [-2(z+ai)^{-3}]_{z=ai} = \frac{1}{4a^3 i}$$

By Cauchy's residue theorem, we have

$$\int_C f(z)dz = \int_{-R}^{R} \frac{dx}{(a^2+x^2)^2} + \int_{\Gamma} \frac{dz}{(a^2+z^2)^2} = 2\pi i \sum R^+ \qquad (1)$$

Now $\lim_{R\to\infty} zf(z) = \lim_{z\to\infty} \frac{z}{(a^2+z^2)^2} = 0$

$\therefore$ $\lim_{R\to\infty} \int_{\Gamma} \frac{dz}{(a^2+z^2)^2} = 0$

and $\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(a^2+x^2)^2} = \int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^2}$

Hence we have from relation (1)

$$\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^2} = 2\int_0^{\infty} \frac{dx}{(a^2+x^2)^2} = 2\pi i . \frac{1}{4a^3 i} = \frac{\pi}{2a^3}$$

or $$\int_0^{\infty} \frac{dx}{(a^2+x^2)} = \frac{\pi}{4a^3}$$

Problem

Evaluate $\int_0^{\infty} \frac{1}{1+x^2} dx$ *using contour integration.*

Answer: π/2

Example

Evaluate $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$

Solution

Consider the integral $\int_C f(z)dz = \int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz$ where C is the contour consisting of a large semi circle Γ of radius R together with real axis from –R to R.

By Cauchy residue theorem, we have



$$\int_C f(z)dz = \int_{-R}^{R} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx + \int_\Gamma \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz = 2\pi i \sum R^+$$

Since $\lim_{R\to\infty} zf(z) = \lim_{z\to\infty} z.\frac{z^2 - z + 2}{z^4 + 10z^2 + 9} = 0$ therefore, we have

$$\lim_{R\to\infty} \int_\Gamma \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz = 0$$

Now $z^4 + 10z^2 + 9 = 0$ or $(z^2 + 9)(z^2 + 1) = 0$ gives poles of f(z).

$\therefore$ $z = \pm i, \pm 3i$ are the simple poles of f(z). Out of these only z = i, 3i are the poles inside C.

Sum of the residues at z = i and 3i

$$= \lim_{z\to i}(z - i) f(z) + \lim_{z\to 3i}(z - 3i) f(z)$$

$$= \frac{i^2 - i + 2}{(i^2 + 9)2i} + \frac{9i^2 - 3i + 2}{6i(9i^2 + 1)} = \frac{1 - i}{16i} + \frac{7 + 3i}{48i} = \frac{5}{24i}$$

Now $\int_{-\infty}^{\infty} f(x)dx = 2\pi i(\text{sum of residues}) = 2\pi i \frac{5}{24i} = \frac{5\pi}{12}$

Problem

Evaluate $\int_0^\infty \frac{dx}{x^4 + 10x^2 + 9}$

Answer: π/24



ASSIGNMENT

Q.1. (PTU 2018, 2 marks): Expand sin z in Taylor's series about the point z = 0.

Answer: A similar problem is solved in this module.

Q.2. (JNTUK 2020, 7 marks): Find Taylor's expansion $f(z) = \frac{z}{(z+1)(z+2)}$ about the point z = -2.

Answer: A similar problem is solved in this module.

Q.3. (UTU 2015, 5 marks): Find the Laurent's series which represent the function $\frac{z^2-1}{(z+1)(z+4)}$ when 1 < |z| < 4.

Q.4. (UTU 2016, 5 marks): Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for 1 < |z| < 3.

Q.5. (UTU 2017, 5 marks): Find the Laurent's series expansion of $\frac{1}{(z+1)(z+2)^2}$ in the region 2 < |z - 1| < 3.

Q.6. (UTU 2013, 2014, 5 marks): Find the Laurent's expansion for $f(z) = \frac{7z-2}{z^3-z^2-2z}$ in the region 1 < |z + 1| < 3.

Answer: Solved in this module.

Q.7. (JNTUK 2021, 2 marks): Define isolated singularity and given an example.

Q.8. (JNTUK 2020, 2 marks): Classify the singularity of $f(z) = (z+1)^2 e^{1/z+1}$ at z = -1.

Q.9. (UTU 2014, 5 marks): Find out the zeros and discuss the nature of the singularities of $f(z) = \frac{z-2}{z^2}\sin\left(\frac{1}{z-1}\right)$.

Answer: z = 0 of a pole of order 2.

Q.10. (JNTU 2019, AKTU 2019, 2 marks): Find the residue of $\frac{z^2}{(z-1)(z-2)^2}$ at pole z = 1 and 2.

Answer: 1, 0

Q.11. (JNTUK 2021, 2 marks): Find the residue of $e^{1/z}$ at z = 0.

Answer: Solved in this module.

Q.12. (JNTUK 2020, 2 marks): Determine the residue of f(z) = z.cos(1/z) at z = 0.

Answer: Solved in this module.

Q.13. (PTU 2018, 2 marks): Find the sum of the residues at each pole of the function f(z) lying inside the circle |z| = 3 where f(z) = tanz/z.



Q.14. (UTU 2016, 5 marks): Find the residue $\frac{z^3}{z^2-1}$ at z = ∞.

Answer: -1

Q.15. (AKTU 2018, 7 marks): State and prove the Cauchy integral formula. Also evaluate $\int_C \frac{1}{(z^2+4)^2}dz = \frac{\pi}{16}$ where C is the circle |z - i| = 2.

Answer: Solved in this module.

Q.16. (UTU 2013): Apply calculus of residues to prove that

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{(a+b)}(a>0, b>0)$$

Q.17. (AKTU 2019, 3.5 marks): Find the residue $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$ at its pole and hence evaluate $\int_C f(z)dz$ where C is the circle |z| = 5/2.

Answer: Solved in this module.

Q.18. (UTU 2013): Determine the poles of the following function and residues at each pole:

$f(z) = \frac{z-1}{(z+1)^2(z-2)}$ and hence evaluate $\oint_C f(z)dz$ where c is a circle |z - i| = 2.

Answer: Poles are -1 (double poles), 2 (simple pole). Residues are -1/9, 1/9.

$\oint_C f(z)dz$ = -2πi/9.

Q.19. (UTU 2014, 5 marks): By Cauchy residue theorem solve the integral $\oint_C \frac{3z^2+z+1}{(z^2-1)(z+3)}dz$ where C: |z| = 2.

Answer: -4πi

Q.20. (UTU 2015, 5 marks): Determine the poles and residue at each pole of the function

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$

Answer: Solved in this module.

Q.21. (UTU 2017, 5 marks): Using Cauchy residue theorem evaluate the integral $\oint_C \frac{z^2-2z}{(z+1)^2(z^2+4)}dz$ where C is the circle |z| = 5.

Answer: Solved in this module.



Q.22. (UTU 2017, 5 marks): Find the value of $\oint_C \frac{z+4}{z^2+2z+5}dz$ where C is |z + 1| = 1.

Answer: 0

Q.23. (JNTUK 2021, 7 marks): Evaluate $\int_C \frac{2e^z}{z(z-3)}dz$ where c: |z| = 2 by residue theorem.

Q.24. (PTU 2019, 10 marks): Use the concept of residues to evaluate $\int_0^{2\pi} \frac{d\theta}{(a-b\sin\theta)}; a > |b|$

Q.25. (PTU 2018, 10 marks): Use the concept of residues to evaluate $\int_0^{2\pi} \frac{dx}{5-4\sin x}$

Answer: 2π/3

Q.26. (AKTU 2018, 7 marks): Evaluate using complex integral $\int_0^{\pi} \sin^4\theta d\theta$

Q.27. (UTU 2017, AKTU 2019, 7 marks): Using complex integration method, evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}$.

Answer: Solved in this module.

Q.28. (AKTU 2020, 7 marks): Evaluate the integration $\int_0^{\pi} \frac{d\theta}{3+2\cos\theta}$.

Q.29. (AKTU 2021, 7 marks): Evaluate the contour integration $\int_0^{2\pi} \frac{1}{a+b\sin\theta}d\theta$ where a > |b|.

Answer: $\frac{2\pi}{\sqrt{a^2-b^2}}$ if a > | b |

Q.30. (JNTUK 2020, 7 marks): Use the method of contour integration to prove that

$$\int_0^{\pi} \frac{\cos 2\theta d\theta}{1+a^2-2a\cos\theta} = \frac{\pi a^2}{1-a^2}; -1 < a < 1.$$

Q.31. (UTU 2015, 5 marks): Use residue calculus to evaluate $\int_0^{2\pi} \frac{1}{5-4\sin\theta}d\theta$

Q.32. (UTU 2017, 5 marks): Evaluate the following integral $\int_0^{\pi} \frac{ad\theta}{a^2+\sin^2\theta}$

Q.33. (JNTUK 2021, 7 marks): Show by the method of contour integral evaluate $\int_0^{\infty} \frac{\cos mx}{(x-a)}dx$

