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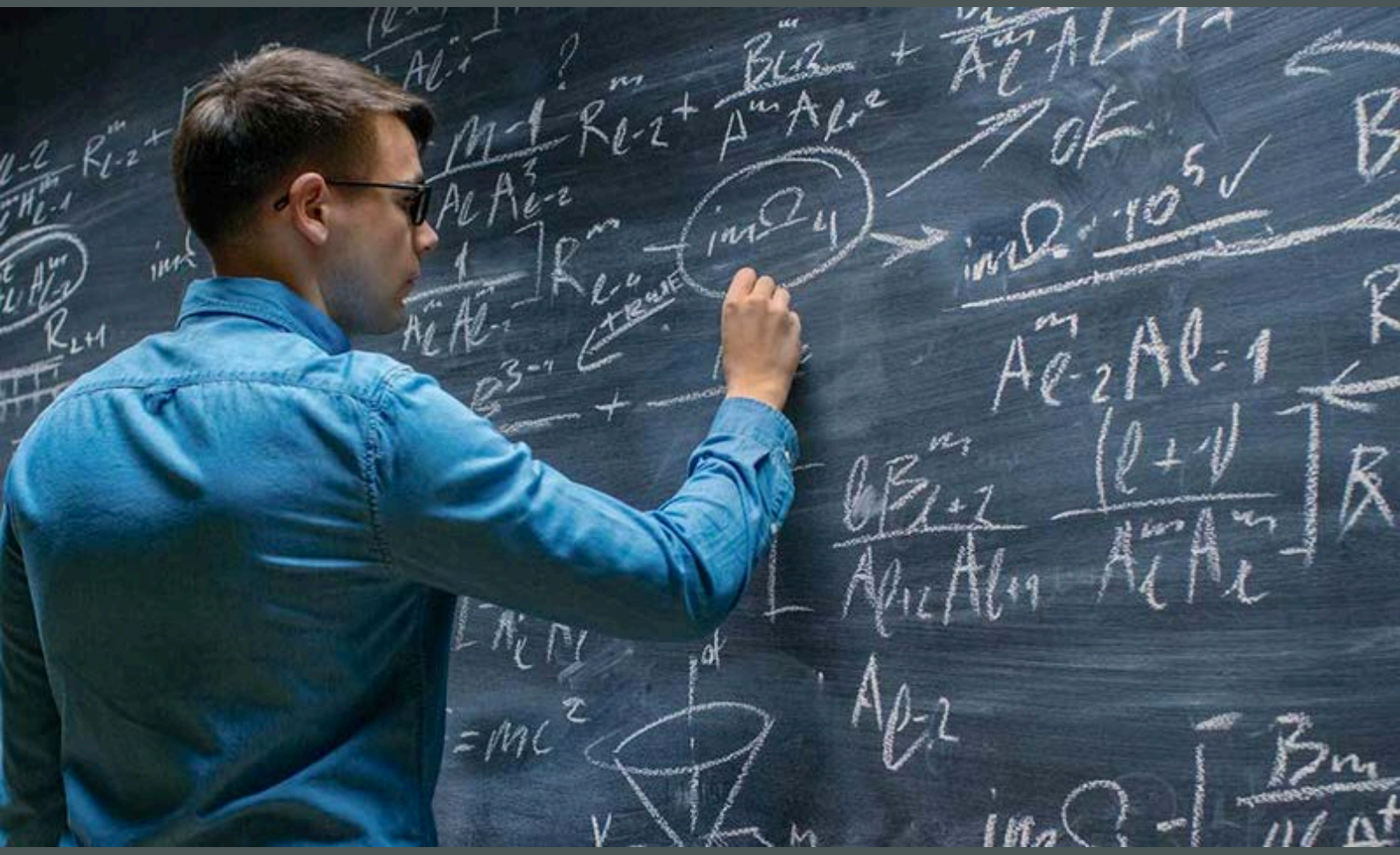

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COMPLEX ANALYSIS - I

REAL NUMBER SYSTEM

- Natural Numbers 1,2,3,4……
- Negative integers and zero -1, -2, -3, … and 0
- Rational numbers 3/4, -8/3, ……
- Irrational numbers $\sqrt{2}$, π etc.

COMPLEX NUMBER SYSTEM

There is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$. To permit solutions of this and similar equations, the set of complex numbers is introduced.

We can consider a complex number as having the form a + bi where a and b are real numbers and i, which is called the imaginary unit, has the property that $i^2 = -1$. If z = a + bi, then a is called the **real** part of z and b is called **imaginary** part of z and are denoted by Re{z} and Im{z} respectively. The symbol z, which can stand for any of a set of complex numbers, is called a *complex variable*.

The **complex conjugate**, or briefly conjugate, of a complex number a + bi is a - bi. The complex conjugate of a complex number z is often indicated by $\bar{z}$ or z*.

Complex conjugate of a + bi is also indicated as $\overline{a+bi}$ which is equal to a - bi.

Example

Find real numbers x and y such that 3x + 2iy -ix + 5y = 7 + 5i.

Solution

The given equation can be written as 3x + 5y +i(2y - x) = 7 + 5i. The equating real and imaginary parts, 3x + 5y = 7, 2y - x = 5. Solving we get x = -1, y = 2.

Example

$$\frac{a+bi}{c+di} = ?$$

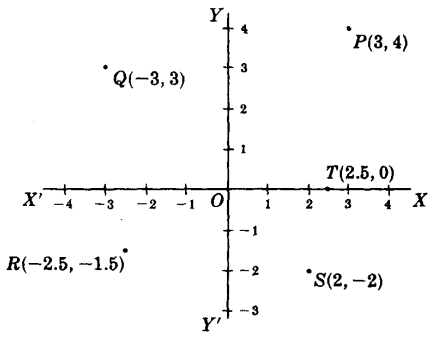
Solution

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} = \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2} = \frac{ac+bd+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$



GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS

Since a complex number x + iy can be considered as an ordered pair of real numbers, we can represent such numbers by points in an xy plane called the complex plane or **Argand diagram**. The complex number represented by P, for example, could then be read as either (3,4) or 3 + 4i.



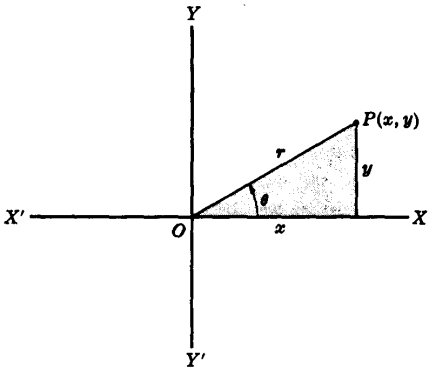
Sometimes we refer to the x and y axes as the real and imaginary axes respectively and to the complex plane as the z plane. The distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the complex plane is given by $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

POLAR FORM OF COMPLEX NUMBERS

If P is the point in the complex plane corresponding to the complex number (x,y) or x + iy, then we see from figure that

$$x = r\cos\theta$$

and $$y = r\sin\theta$$



where $r = \sqrt{(x2 + y2)} = |x + iy|$ is called the modulus or absolute value of z = x + iy[denoted by z or |z|]; and θ called the amplitude or argument of z = x + iy[denoted by arg z], is the angle which line OP makes with the positive x axis.

It follows that

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

which is called the polar form of the complex number, and r and θ are called polar coordinates. It is sometimes convenient to write the abbreviation cisθ for cosθ + isinθ.

Example

Express $2 + 2\sqrt{3}i$ in polar form.

Solution

Modulus of absolute value, $r = |2 + 2\sqrt{3}i| = \sqrt{(4 + 12)} = 4$

Amplitude or argument $\theta = \sin^{-1}2\sqrt{3}/4 = \sin^{-1}\sqrt{3}/2 = 60^0 = \pi/3$ (radians)

Then $2 + 2\sqrt{3}i = r(\cos\theta + i\sin\theta) = 4(\cos 60^0 + i\sin 60^0) = 4(\cos\pi/3 + i\sin\pi/3)$

Note *this*: The result can also be written as $4\text{cis}\pi/3$ or using Euler's formula(wil be discussed later), as $4e^{\pi i/3}$.

DERIVATIVES

If f(z) is single values in some region R of the z plane, the derivative of f(z) is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Note: Sometimes h is used instead of z.

ANALYTIC FUNCTIONS

If the derivative f'(z) exists at all points z of a region R, then f(z) is said to be analytic in R and is referred to as an analytic function in R. This function is also known as a regular or **holomorphic**

function.

A function f(z) is said to be analytic at a point z_0 if there exists a neighbourhood $|z - z_0| < \delta$ at all points of which f'(z) exists.

CAUCHY RIEMANN (CR) EQUATION IN CARTESIAN COORDINATES

A necessary condition that w = f(z) = u(x, y) + iv(x, y) be analytic in a region R is that, in R, u and v satisfy the Cauchy Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad [\text{or } u_x = v_y]$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad [\text{or } u_y = -v_x]$$

Also $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Cauchy–Riemann Equation of the Function f (z) = u(x, y) + iv(x, y) at Origin

$$\frac{\partial u}{\partial x} = Lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$\frac{\partial u}{\partial y} = Lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$\frac{\partial v}{\partial x} = \underset{x\to 0}{Lim}\frac{v(x,0)-v(0,0)}{x}$$

$$\frac{\partial u}{\partial y} = \underset{y\to 0}{Lim}\frac{v(0,y)-v(0,0)}{y}$$

If at z = 0, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Then C–R equations are satisfied at z = 0.

Example

Prove that the function f(z) = $z^2 = x^2 - y^2 + 2ixy$ is analytic for all z, f'(z) = 2z.

Solution

We have u = $x^2 - y^2$, v = 2xy,

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x$$

and the Cauchy-Riemann equations are satisfied for all x, y. Hence, f(z) is analytic.

Example (UTU 2013, 2018, 5 marks)

Find the values of c_1 and c_2 such that the function

$$f(z) = x^2 + c_1y^2 - 2xy + i(c_2x^2 - y^2 + 2xy)$$

is analytic. Also find f'(z).

Solution

Equating real and imaginary parts, we get

$$u = x^2 + C_1y^2 - 2xy; v = C_2x^2 - y^2 + 2xy$$

$\therefore$ $\frac{\partial u}{\partial x} = 2x - 2y; \frac{\partial v}{\partial x} = 2C_2x + 2y$

and $\frac{\partial u}{\partial y} = 2C_1y - 2x; \frac{\partial v}{\partial y} = -2y + 2x$

C-R equations are

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}&=-\frac{\partial v}{\partial x}\end{aligned}\right]$$

It implies that

$$2x - 2y = -2y + 2x \qquad (1)$$

and $$2C_1y - 2x = -2C_2x - 2y \qquad (2)$$

From (2) equating the coefficient of x and y.

$C_1 = -1$ and $C_2 = 1$.

On putting these values

$$\frac{\partial u}{\partial x}=2x-2y;\frac{\partial v}{\partial x}=2x+2y$$

Now $$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=(2x-2y)+i(2x+2y)=2(1+i)(x+iy)=2(1+i)z$$

Example

Prove that $\frac{d}{dz}(z^2\bar{z})$ *does not exist.*

Solution

From Cauchy Riemann conditions

$$z^2\bar{z}=(x+iy)^2(x-iy)=\left[x(x^2-y^2)+2xy^2\right]+i\left[2x^2y+y(y^2-x^2)\right]$$

So $$u=x(x^2-y^2)+2xy^2$$

$$v=2x^2y+y(y^2-x^2)$$

$$u_x=3x^2-y^2+2y^2$$

$$u_y=-2xy+4xy$$

$$v_x=4xy-2xy$$

$$v_y=2x^2+3y^2-x^2$$

CR conditions are not satisfied for any x, y.

So, $f'=u_x+iv_x=v_y-iu_y$ does not exist for any x, y i.e. for any z.



Example

Determine where the Cauchy Riemann equations are satisfied for the given functions. Determine the region of analytically.

(i) $f(z) = e^z = e^x(\cos y + i \sin y)$

(ii) $f(z) = (x-y)^2 + 2i(x+y)$

(iii) $f(z) = e^{i\bar{z}} = e^y(\cos x + i \sin x)$

(iv) $f(z) = \cos x(\cosh y + a \sinh y) + i \sin x(\cosh y + b \sinh y)$ where a and b are constants.

(v) $f(z) = \dfrac{(z+3i)^5}{(z^2-2z+5)^2}$

Solution

(i) We have

$$u(x, y) = e^x \cos y$$
$$v(x, y) = e^x \sin y$$
$$u_x = e^x \cos y$$
$$u_y = -e^x \sin y$$
$$v_x = e^x \sin y$$
$$v_y = e^x \cos y$$

So $u_x = v_y \; and \; u_y = -v_x$

i.e. CR conditions are satisfied for all x and y. Therefore given function e^z is analytic everywhere. Thus e^z is an entire function.

(ii) We have

$$u = (x-y)^2$$
$$v = 2(x+y)$$
$$u_x = 2(x-y)$$
$$u_y = -2(x-y)$$
$$v_x = 2$$
$$v_y = 2$$

So $u_x = 2(x-y) = v_y = 2$ if $(x-y) = 1$



Also $u_y = -2(x-y) = -v_x = -2 \; if \; (x-y) = 1$

Thus CR equations are satisfied only along the straight lines x – y = 1.

So $f'(z) = u_x + iv_x = 2(x-y) + i2 = 2 + 2i$ exists only along the line $x - y = 1$, not through any region (neighbourhood) R. Hence f(z) is nowhere analytic.

(iii) We have

$$u = e^y \cos x$$
$$v = e^y \sin x$$
$$u_x = -e^y \sin x$$
$$u_y = e^y \cos x$$
$$v_x = e^y \cos x$$
$$v_y = e^y \sin x$$

Thus $u_x \neq v_y \; and \; u_y \neq -v_x$ for any x and y. CR conditions are not satisfied for any z.

Hence $f(z) = e^{\bar{iz}}$ is nowhere analytic.

(iv) We have

$$u = \cos x(\cosh y + a \sinh y)$$
$$v = \sin x(\cosh y + b \sinh y)$$
$$u_x = -\sin x(\cosh y + a \sinh y)$$
$$v_y = \sin x(\sinh y + b \cosh y)$$
$$u_y = \cos x(\sinh y + a \cosh y)$$
$$v_x = \cos x(\cosh y + b \sinh y)$$

So $u_x = v_y \; and \; u_y = -v_x$ if a = b = -1

Hence f is analytic when a = b = -1.

(v) We have

$$f(z) = \frac{g(z)}{h(z)}$$

where $g(z) = (z + 3i)^5$ (polynomial) is analytic everywhere and $h(z) = (z^2 - 2z + 5)^2$ (polynomial) is analytic everywhere.

The quotient f(z) = g(z)/h(z) is analytic everywhere except when $h(z) = (z^2 - 2z + 5)^2 = 0$ Thus f(z) is analytic everywhere except at $(z^2 - 2z + 5)^2 = 0$ i.e. at $z = 1 \pm 2i$.



Problem

Are the following functions analytic?

a. $f(z) = z^3 + z$

b. $f(z) = \bar{z}$

c. $f(z) = \dfrac{1}{(1-z)}$

d. $f(z) = \dfrac{1}{z^2}$

e. $f(z) = x^2 + iy^3$

Answer: (a) Yes, (b) No, (c) Yes, except at z = 1, (d) Yes, except at z = 0 (e) nowhere analytic (CR satisfied at origin)

Example (JNTUK 2021)

Show that the function f (z) = $|z|^2$ is analytic at z = 0, although the Cauchy–Riemann equations are satisfied at that point.

Solution

Given $f(z) = |z|z\bar{z} = (x+iy)(x-iy) = x^2 + y^2$

$\therefore$ $f(z) = u + iv = x^2 + y^2$

$\therefore$ $u = x^2 + y^2$ and $v = 0$

At z = 0 (i.e. at origin)

$$\frac{\partial u}{\partial x} = \underset{x\to 0}{Lim}\frac{u(x,0)-u(0,0)}{x} = \underset{x\to 0}{Lim}\frac{x^2-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y\to 0}{Lim}\frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{Lim}\frac{y^2-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x\to 0}{Lim}\frac{v(x,0)-v(0,0)}{x} = \underset{x\to 0}{Lim}\frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y\to 0}{Lim}\frac{v(0,y)-v(0,0)}{y} = \underset{y\to 0}{Lim}\frac{0-0}{y} = 0$$

$\therefore$ at z = 0, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$



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and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, C–R equations are satisfied at the origin.

Also $$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{(x^2 + y^2) - 0}{x + iy}$$

Let $z \to 0$ along the line $y = mx$, then

$$f'(0) = \lim_{x\to 0} \frac{x^2 + m^2x^2}{x + imx} = \lim_{x\to 0} \frac{(1+m^2)x}{(1+m)} = 0$$

Therefore, f'(0) is unique. Hence, the function $f(z) = |z|^2$ is analytic at z = 0.

Example (UTU 2015, JNTUK 2021, AKTU 2018, 7 marks)

Show that the function $f(z) = |xy|^{1/2}$ is not analytic at the origin, although the Cauchy–Riemann equations are satisfied at that point.

Or

Give an example of a function in which Cauchy Riemann Equations are satisfied yet the function is not analytic at the origin. Justify your answer.

Solution

Given $f(z) = u + iv = |xy|^{1/2}$

$\therefore$ $u(x, y) = |xy|^{1/2}$, $v(x, y) = 0$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{(0-0)}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{(0-0)}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{(0-0)}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = 0$$

$\therefore$ at z = 0, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$



Then C–R equations are satisfied at z = 0.

Now $$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{|xy|^{1/2} - 0}{x + iy}$$

If $z \to 0$ along the path $y = mx$, then we have

$$f'(0) = \lim_{x\to 0} \frac{|mx^2|^{1/2}}{(x + mix)} = \lim_{x\to 0} \frac{|m|^{1/2}}{(1 + im)}$$

Limits depends on m. So f'(z) is not unique at origin. Thus f'(0) does not exist. Hence f (z) is not analytic at origin.

Example (UTU 2013, 2017, 5 marks)

Show that the function f(z) is defined by $f(z) = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}}, z \neq 0, f(0) = 0$ *is not analytic at the origin even though it satisfies Cauchy-Riemann equation at the origin.*

Solution

Equating real and imaginary parts, we get

$$u = \frac{x^3 y^5}{x^4 + y^{10}}; v = \frac{x^2 y^6}{x^4 + y^{10}}$$

Now $$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0 + h, 0) - u(0, 0)}{h} = \lim_{h\to 0} \frac{(0 / h^4)}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0, 0 + k) - u(0, 0)}{k} = \lim_{k\to 0} \frac{(0 / k^{10})}{k} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(0 + h, 0) - v(0, 0)}{h} = \lim_{h\to 0} \frac{(0 / h^4)}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{v(0, 0 + k) - v(0, 0)}{k} = \lim_{k\to 0} \frac{(0 / k^{10})}{k} = 0$$

$\therefore$ at z = 0, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Then C–R equations are satisfied at z = 0.



But $$f'(0)=\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{x^2y^5(x+iy)}{x^4+y^{10}}-0\right].\frac{1}{x+iy}\text{ (increment z)}$$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2y^5}{x^4+y^{10}}$$

let $z\to 0$ along the line y = mx, then

$$f'(0)=\lim_{x\to 0}\frac{m^5x^7}{x^4+m^{10}x^{10}}=\lim_{x\to 0}\frac{m^5x^3}{1+m^{10}x^6}=\frac{0}{1}=1 \qquad (1)$$

Again let $z\to 0$ along the curve $y^5 = x^2$

From $$f'(0)=\lim_{x\to 0}\frac{x^4}{x^4+x^4}=\frac{1}{2} \qquad (2)$$

(I) and (2) shows that f'(0) does not exist. Hence, f(z) is not analytic at origin although Cauchy-Riemann equations arc satisfied there.

Example (JNTUH 2012, 2013, 2014, UTU 2017, 5 marks)

If f(z) is regular function of z, show that $\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}|f(z)|^2=4|f'(z)|^2$

Solution

Given that f(z) is an analytic function.

$\therefore$ $$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y};\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$|f(z)|^2 = u^2 + v^2 = \phi$ (say)

$$\frac{\partial \phi}{\partial x}=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2}=2u\frac{\partial^2 u}{\partial x^2}+2\left(\frac{\partial u}{\partial x}\right)^2+2v\frac{\partial^2 v}{\partial x^2}+2\left(\frac{\partial v}{\partial x}\right)^2$$

and $$\frac{\partial \phi}{\partial y}=2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}$$

$$\frac{\partial^2 \phi}{\partial y^2}=2u\frac{\partial^2 u}{\partial y^2}+2\left(\frac{\partial u}{\partial y}\right)^2+2v\frac{\partial^2 v}{\partial y^2}+2\left(\frac{\partial v}{\partial y}\right)^2$$



$$\therefore \quad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=2u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+2v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$
$$+2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]+2\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}$$

$$\therefore \quad |f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2$$

$$\therefore \quad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=2|f'(z)|^2+2|f'(z)|^2=4|f'(z)|^2 \quad \left[\because \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0;\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0\right]$$

HARMONIC FUNCTIONS

If the second partial derivatives of u and v with respect to x and y exist and are continuous in a region R then

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

and

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

It follows that under these conditions the real and imaginary parts of an analytic function satisfy Laplace's equation.

Functions such as u(x, y) and v(x, y) which satisfy Laplace's equation in a region R are called harmonic functions.

Conjugate Harmonic Function

The real part u of an analytic function $f=u+iv$ is known as the conjugate harmonic function of v and vice versa (i.e. v is the conjugate harmonic of u).

Example

Is the function $u(x,y)=2xy+3xy^2-2y^3$ *harmonic?*



Solution

We have
$$u_x = 2y + 3y^2$$
$$u_{xx} = 0$$
$$u_y = 2x + 6xy - 6y^2$$
$$u_{yy} = 6x - 12y$$

So $u_{xx} + u_{yy} \neq 0$

Therefore u is not harmonic.

Example

Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

Solution

$$\frac{\partial u}{\partial x} = (e^{-x})(\sin y) + (-e^{-x})(x\sin y - y\cos y) = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y\right)$$

$$= -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y \qquad (1)$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y) = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y)$$

$$= -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y \qquad (2)$$

Adding (1) and (2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and u is harmonic.}$$

Problem

Verify that the given function is harmonic and find its conjugate harmonic function,

(i) $u = x^2 - y^2 - y$

(ii) $u = 3x^3y + 2x^2 - y^3 - 2y^2$

Answer:



(i) $v = 2xy + x + c,\ f(z) = z^2 + iz + c$

(ii) not harmonic

Example (JNTUK 2004, 2005, 2020, 7 marks)

If f(z) is an analytic function show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^2 = 2|f'(z)|^2$$

Solution

Let f(z) = u + iv i.e. Re f(z) = u

Also f'(z) = $u_x + iv_x$ so that $|f'(z)| = \sqrt{u_x^2 + v_x^2}$

Squaring $$|f'(z)|^2 = u_x^2 + v_x^2 \quad (1)$$

Now $$\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x} \Rightarrow \frac{\partial^2}{\partial x^2}(u^2) = 2[u_x^2 + uu_x] \quad (2)$$

Similarly $$\frac{\partial^2}{\partial y^2}(u^2) = 2[u_y^2 + uu_{yy}] \quad (3)$$

Now Eqs. (2) + (3) gives

$$\nabla^2(u^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = 2[u_x^2 + u_y^2 + u(u_{xx} + u_{yy})] \quad (4)$$

We know that Re f(z) = u satisfies the Laplace's equation so that

$$\nabla^2(u^2) = 2(u_x^2 + u_y^2) = 2|f'(z)|^2 \text{ by equation (1)}$$

$\Rightarrow$ $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^2 = 2|f'(z)|^2$$

CAUCHY RIEMANN EQUATIONS IN POLAR COORDINATES

Let $x = r\cos\theta; y = r\sin\theta$

Then $z = x_iy = r\cos\theta + ir\sin\theta = re^{i\theta}$

So $u + iv = f(z) = f(re^{i\theta})$

Differentiating partially w.r.t. r and θ



$$\frac{\partial u}{\partial r}+i\frac{\partial u}{\partial r}=f'(re^{i\theta}).e^{i\theta}$$

$$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}=f'(r.e^{i\theta}).ire^{i\theta}=ir\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)$$

Equating the real and imaginary parts

$$\frac{\partial u}{\partial r}=\frac{1}{r}.\frac{\partial v}{\partial \theta}$$

and $$\frac{\partial v}{\partial r}=-\frac{1}{r}.\frac{\partial u}{\partial \theta}$$

The derivative f' can be calculated using

$$f'=e^{-i\theta}(u_r+iv_r)$$

or $$f'=\frac{-i}{re^{i\theta}}(u_\theta+iv_\theta)$$

Example

Find analytic function f(z) = u(r, θ) + iv(r, θ) such that v(r, θ) = $r^2\cos 2\theta$ - $r\cos\theta$ + 2.

Solution

Cauchy-Riemann equations in polar coordinates are given by

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$$

and $$\frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

$\therefore$ $$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}=-2r\sin 2\theta+\sin\theta \qquad (1)$$

and $$\frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}=-2r^2\cos 2\theta+r\cos\theta \qquad (2)$$

Integrating equation (1) wrt r, we get

$$u = -r^2\sin 2\theta + r\sin\theta + \phi(\theta)$$

where $\phi(\theta)$ is an arbitrary function.

This leads to $$\frac{\partial u}{\partial \theta}=-2r^2\cos 2\theta+r\cos\theta+\phi'(\theta) \qquad (3)$$



From (2) and (3)

$$-2r^2\cos2\theta + r\cos\theta = -2r^2\cos2\theta + r\cos\theta + \phi'(\theta)$$

$\therefore\ \phi'(\theta) = 0 \Rightarrow \phi(\theta) = c$, where c is any arbitrary constant.

Thus $u = -r^2\sin2\theta + r\sin\theta + c$

$\therefore$ $f(z) = u + iv = r^2(-\sin2\theta + i\cos2\theta) + r(\sin\theta - i\cos\theta) + c + 2i$

Problem

Find the analytic function f(z) = u(r, θ) + iv(r, θ) such that $v(r,\theta) = \left(r - \frac{1}{r}\right)\sin\theta, r \neq 0$

Answer: $f(z) = u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta + c$

Example

Find the analytical function whose real part is $r^2\cos2\theta + r\sin\theta$.

Solution

Let $f(z) = u + iv$ where $u = r^2\cos2\theta + r\sin\theta$.

Then $\frac{\partial u}{\partial r} = 2r\cos2\theta + \sin\theta$

and $\frac{\partial u}{\partial \theta} = -2r^2\sin2\theta + r\cos\theta$

By total differentiation formula and using CR equations $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}; \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

$$dv = \frac{\partial v}{\partial r}dr + \frac{\partial v}{\partial \theta}d\theta = \left(\frac{-1}{r}.\frac{\partial u}{\partial \theta}\right)dr + \left(r\frac{\partial u}{\partial r}\right)d\theta$$

$$= -\frac{1}{r}(-2r^2\sin2\theta + r\cos\theta)dr + r(2r\cos2\theta + \sin\theta)d\theta$$

$$= (2r\sin2\theta - \cos\theta)dr + (2r^2\cos2\theta + r\sin\theta)d\theta$$

$$= d(r^2\sin2\theta) - d(r\cos\theta) \quad \text{[Exact differential equation]}$$

Integrating, we get

$$v = r^2\sin2\theta - r\cos\theta + c$$

$\therefore$ $f(z) = u + iv = (r^2\cos2\theta + r\sin\theta) + i(r^2\sin2\theta - r\cos\theta + c)$



$= r^2(\cos 2\theta + i\sin 2\theta) - ir(\cos\theta + i\sin\theta) + ic$

This is required analytic function.

Milne Thompson Method

Step 1: If u is given, take $f' = u_x - iu_y$

If v is given take $f' = v_y + iv_x$

Step 2: Replace x by z and y by 0 in f'

Step 3: Integrate f' w.r.t. z.

Example (AKTU 2021, 7 marks)

Determine the analytic function f(z) = u + iv in terms of whose real part is e^x(xcosy - ysiny).

Solution

$$u = e^x(x\cos y - y\sin y)$$

$$\therefore \quad \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$$

and $$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$$

Now $f(z) = u + iv$

$\therefore$ $f'(z) = u_x + ix_x = u_x - iu_y$

$= e^x(x\cos y - y\sin y + \cos y) + ie^x(x\sin y + y\cos y + \sin y)$

By Milne Thomson method, put x = z and y = 0 to express f'(z) in terms of z, we get

$f'(z) = e^z(z + 1)$

After integration

$f(z) = ze^z + ic$

Here we take constant of integration as imaginary since u does not contain any constant.

Example

Find the analytic function f(z) = u + iv

where $u = e^x(x\cos y - y\sin y) + 2\sin x.\sinh y + x^3 - 3xy^2 + y$.



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Solution

Method 1

Differentiating partially w.r.t. x and y

$$u_x = e^x(x\cos y - y\sin y) + e^x(\cos y) + 2\cos x\sinh y + 3x^2 - 3y^2$$

$$u_y = e^x(-x\sin y - \sin y - y\cos y) + 2\sin x\cosh y - 6xy + 1$$

We know that $f'(z) = u_x + iv_x = u_x - iu_y$

Replace x by z and y = 0, then

$$f'(z) = e^z(z.1 - 0) + e^z(1) + 0 + 3z^2 - 0 - 2i\sin z - i$$

Integrating w.r.t. z, we get

$$f(z) = \int (ze^z + e^z + 3z^2 - 2i\sin z - i)dz$$

$$f(z) = ze^z - e^z + e^z + z^3 + 2i\cos z - iz + c$$

Method 2

We have $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial y}dy \quad \text{[by CR equations]}$$

$$dv = -e^x(-x\sin y - y\cos y - \sin y)dx + e^x(x\cos y - y\sin y + \cos y)dy$$

Integrating both sides, we get

$$v = \int e^x(x\sin y + y\cos y + \sin y)dy + \int (\text{terms not containing x})dy + C$$

$$= \sin y\int xe^x dx + (y\cos y + \sin y)\int e^x dx + \int 0.dy + C$$

$$= \left[(x-1)\sin y + y\cos y + \sin y\right]e^x + C$$

$v(x, y) = (x\sin y + y\cos y)e^x + C$

$\therefore$ $f(z) = u(x, y) + iv(x, y)$

$= e^x(x\cos y - y\sin y) + i[e^x(x\sin y + y\cos y) + C]$

$= xe^x(\cos y = i\sin y) + iye^x(\cos y + i\sin y) + iC$

Problem

Determine the analytic function w = u + iv if $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.



Answer: By Milne method: $w = z^3 + 3z^2 + c$

By alternative method: $w = u + iv = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y + 6xy - y^3 + c)$

Example (PTU 2019, 5 marks)

If $u+v=\dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$ *and f(z) = u + iv is an analytic function of z = x + iy, find f(z) in terms of z.*

Solution

Given u + iv = f(z)

$\therefore$ iu - v = if(z)

Adding these, we get

$$(u - v) + i(u + v) = (1 + i)f(z) = F(z) \quad \text{[say]}$$

Now, F(z) will be analytic because f(z) is analytic.

Let U = u - v

and V = u + v

Then F(z) = U + iV [Analytic function]

Now $$V = u+v = \frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x} = \frac{2\sin 2x}{2\cosh 2y-2\cos 2x}$$

$\therefore$ $$\frac{\partial V}{\partial y} = -\frac{2\sin 2x\sinh 2y}{(\cosh 2y-\cos 2x)^2} = \phi_1(x,y)$$

and $$\frac{\partial y}{\partial x} = \frac{2\cos 2x(\cosh 2y-\cos 2x)-2\sin^2 2x}{(\cosh 2y-\cos 2x)^2} = \phi_2(x,y)$$

Using Milne's method

$$F'(z) = \phi_1(z,0) + i\phi_2(z,0)$$

$$= 0 + i\frac{2\cos 2z(1-\cos 2z)-2\sin^2 2z}{(1-\cos 2z)^2} = -2i\frac{1}{1-\cos 2z} = -i\cos ec^2 z$$

Integrating it, we get

$$F(z) = -i\int \cos ec^2 z dz + c = i\cot z + c$$

or (1 + i)f(z) = icotz + c [because F(z) = (1 + i)f(z)]



$$\therefore \qquad f(z) = \frac{i}{1+i}\cot z + \frac{c}{1+i} = \frac{1}{2}(1+i)\cot z + c_1$$

Problem

if f(z) = u + iv is an analytic function of z and

$$u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$$

Find f(z) subject to condition f(π/2) = 0.

Solution

u + iv = f(z)

iu - v = if(z)

Adding (u - v) +i(u + v) = (1 + i)f(z) = F(z) [say]

i.e. u - v = U

and u + v = V

Then U + iV = F(z) [Analytic function]

Now $$U = \frac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y} = \frac{1}{2}\left[1 + \frac{\sin x + \sinh y}{\cos x - \cosh y}\right]$$

$$\therefore \qquad \frac{\partial U}{\partial x} = \frac{1}{2}\frac{1 + \sin x \sinh y - \cos x \cosh y}{(\cos x - \cosh y)^2} = \phi_1(x, y)$$

and $$\frac{\partial U}{\partial y} = \frac{1}{2}\frac{-1 + \sin x \sinh y + \cos x \cosh y}{(\cos x - \cosh)^2} = \phi_2(x, y)$$

By Milne's method, we have

$$F'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

$$= \frac{1}{2}.\frac{1}{1-\cos z} + \frac{i}{2}.\frac{1}{1-\cos z} = \frac{1}{4}(1+i)\cos ec^2(z/2)$$

Now, integrating it, we get

$$F(z) = -\frac{1}{2}(1+i)\cot\left(\frac{z}{2}\right) + c$$

i.e. $$(1+i)f(z) = -\frac{1}{2}(1+i)\cot\left(\frac{z}{2}\right) + c \qquad [F(z) = (1 + i)f(z)]$$



or $$f(z) = -\frac{1}{2}\cot\left(\frac{z}{2}\right) + c_1$$

Now z = π/2, f(π/2) = 0

$\therefore$ $c_1 = 1/2$

$\therefore$ $$f(z) = \frac{1}{2}[1 - \cot(z/2)]$$

Problem

if u - v = (x - y)(x² + 4xy + y²) and f(z) = u + iv is an analytic function of z = x + iy, find f(z) in terms of z.

Answer: f(z) = iz³ + c₁

CAUCHY'S INTEGRAL THEOREM

If f(z) is an analytic function and f ¢(z) is continuous at each point inside and on a closed curve C then

$$\oint_C f(z)dz = 0$$

If f (z) is analytic in the region R between two simple closed curves C_1 and C_2 then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

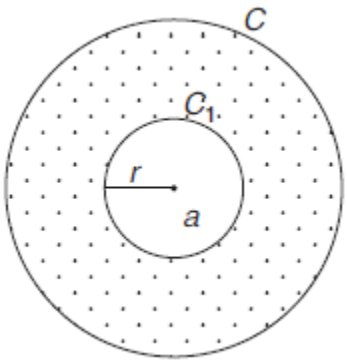
CAUCHY'S INTEGRAL FORMULA

If f(z) is analytic inside and on a simple closed curve C and "a" is any point inside C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

where C is traversed in the positive (counterclockwise) sense.





CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION

If a function f (z) is analytic in a region R then its derivative at any point z = a of R is also analytic in R and is given by

$$f'(a) = \frac{1}{2\pi i}\oint_0 \frac{f(z)}{(z-a)^2}dz$$

where C is any closed curve in R surrounding the point z = a.

Also $$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz, \quad n = 1,2,3\ldots.$$

The results of these equations are called Cauchy's integral formulae.

ABSOLUTE VALUE OF COMPLEX NUMBER

$$|a+bi| = \sqrt{a^2+b^2}$$

Example (GTU 2013)

Evaluate $\int_C e^{z^2}dz$ *where C is any closed contour. Justify your answer.*

Solution

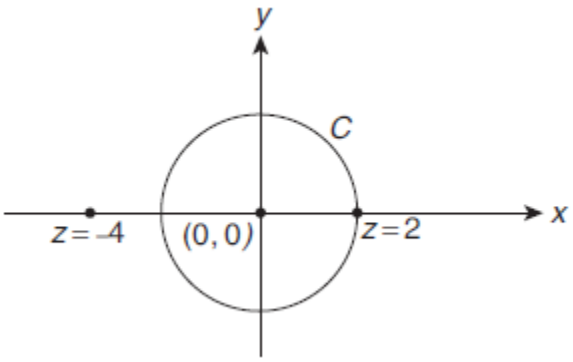
- $f'(z) = e^{z^2}.2z$. f(z) is **differentiable**. Hence z is analytic in the z plane. f'(z) is **continuous** in z plane.
- By Cauchy's integral theorem, $\int_C e^{z^2}dz = 0$

Example

Evaluate $\int_C \frac{dz}{z+4}$ *where C is circle* $|z| = 2$.

Solution

- From f(z) = 1/(z + 4), f(z) is **not analytic at z = 4**.
- From |z| = 0, a circle is drawn shown in figure. We see that z = -4 lies outside the circle.



- f(z) is **analytic inside** and on C. Hence, f'(z) is **continuous** inside C.
- Using Cauchy's theorem, $\int_C \frac{dz}{z+4} = 0$

Problem

Evaluate $\int_C \frac{dz}{(z-3)^2}$ *where C is circle* $|z| = 1$.

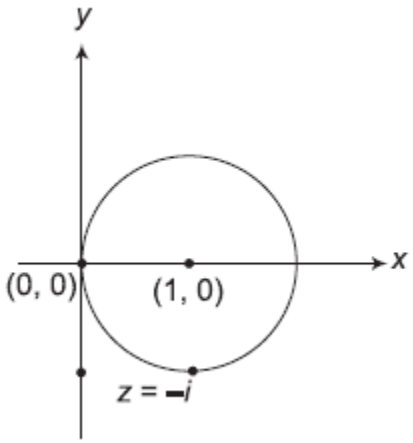
Answer: 0

Example (GTU 2015)

Evaluate $\oint_C \frac{e^z}{z+i} dz$ *where C:* $|z - 1| = 1$.

Solution

Let us draw the circle from given data.



- For z = -i, |z - 1| = |1- i| = √](-1)^2 + (-1)^2] = √2 > 1 $[|a+bi| = \sqrt{a^2+b^2}]$
- Hence, z = -i lies outside C.
- f(z) is **analytic** inside and on C. Hence f'(z) is **continuous** inside C.
- By Cauchy's integral

$$\oint_C \frac{e^z}{z+i} dz = 0$$

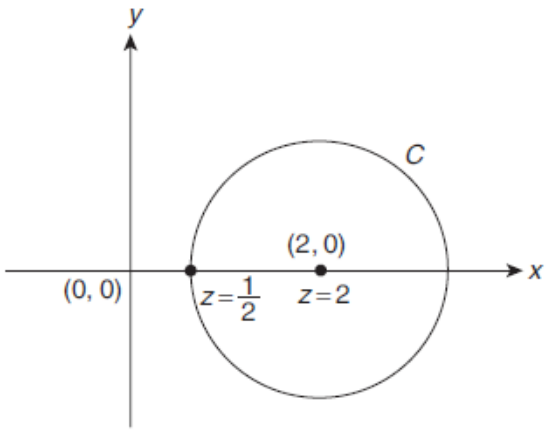
Example

Evaluate $\oint_C \frac{z}{z-2} dz$ *where C is the circle |z - 2| = 3/2.*

Solution

let $I = \oint_C \frac{z}{z-2} dz$

C is the circle |z - 2| = 3/2 with centre (2, 0) and a radius of 3/2 given below.



For z = 2, |z - 2| = |2 - 2| = 0 < 3/2. Hence z = 2 lies *inside* C. Hence Cauchy integral will not be 0.

Let f(z) = z, f(z) is analytic inside and on C.

By cauchy's integral formula

$$\oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

$$\oint_C \frac{z}{z-2}dz = 2\pi i[z]_{z=2} = 4\pi i$$

Problem

Evaluate $\oint_C \frac{e^z}{z-1}dz$ *if C is* $|z| = 2$.

Answer: $2\pi ei$

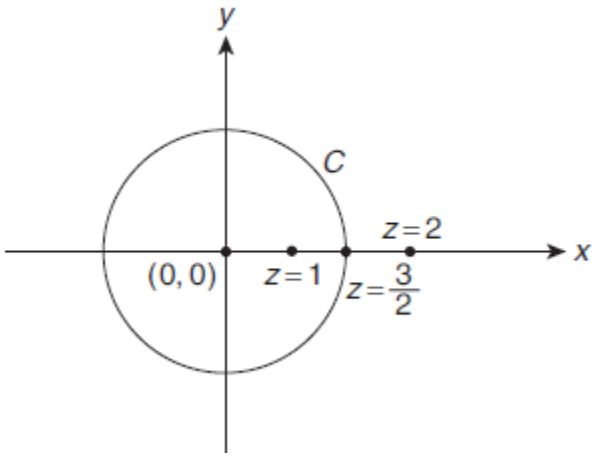
Example (AKTU 2020, 7 marks)

Evaluate $\oint_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz$ *where C is circle* $|z| = 3/2$.

Solution

$$I = \oint_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz$$

Circle is given below.



For z = 1, |z| = |1| = 1 <3/2. Hence z = 1 lies inside C. Hence Cauchy value can not be zero.

For z = 2, |z| = |2| = 2 > 3/2. Hence z lies outside C.

Let $f(z)=\dfrac{\cos \pi z^2}{z-2}$

f(z) is analytic inside and on C.

Now $\oint_C \dfrac{\cos \pi z^2}{(z-1)(z-2)} = \oint_C \dfrac{\dfrac{\cos \pi z^2}{z-2}}{z-1} dz$

☞ *(z-1) is kept in denominator as z = 1 lies inside C*

By Cauchy's integral formula

$$\oint_C \frac{f(z)}{z-a} = 2\pi i f(a)$$

$\therefore$ $$\oint \frac{\dfrac{\cos \pi z^2}{z-2}}{z-1} dz = 2\pi i \left[\frac{\cos \pi z^2}{z-2}\right]_{z=1} = 2\pi i \left[\frac{\cos \pi}{1-2}\right] = 2\pi i$$

Example (AKTU 2021, 7 marks)

State and prove the Cauchy integral formula. Also evaluate $\int_C \dfrac{4-3z}{z(z-1)(z-2)} dz$ *where C is circle |z| = 3/2.*

Solution

The poles z = 0 and z = 1 lie inside the circle | z | = 1.5. The pole z = 2 lies outside the circle |z| = 1.5.

Now write the given expression in such a manner that in one part z comes as denominator and in other part (z - 1) comes in denominator. It is because z = 0 and z = 1 lies inside the circle |z| = 1.5.

$$\therefore \quad \oint_C \frac{4-3z}{z(z-1)(z-2)} = \oint_{C_1} \frac{\frac{4-3z}{(z-10(z-20}}{z} dz + \oint_{C_2} \frac{\frac{4-3z}{z(z-2)}}{z-1} dz$$

Using Cauchy's integral formula

$$\oint_C \frac{4-3z}{z(z-1)(z-2)} = 2\pi i \left[\frac{4-3z}{(z-10(z-20}\right]_{z=0} + 2\pi i \left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

$$= 2\pi i \left[\frac{4}{(-1)(-2)}\right] + 2\pi i \left[\frac{4-3}{1-2}\right] = 2\pi i$$

Example (JNTUK 2021, 7 marks)

Evaluate $\int_C \frac{ze^z}{(z-a)^3} dz$ *where "a" lies within a closed curve by Cauchy integral formula.*

Solution

Using $f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$

$$f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} dz$$

But $f'(a) = ze^z + e^z$

$f''(a) = ze^z + 2e^z$

$\therefore \quad f''(a) = (a + 2)e^a$

$$\therefore \quad \oint_C \frac{ze^z}{(z-a)^3} dz = \pi i(a+2)e^a$$



Problem

Evaluate $\oint_C \frac{z}{(z-1)^3} dz$ *where C:* $|z| = 2$.

Answer: 0

Example (AKTU 2019, 3.5 marks)

Verify Cauchy theorem by integrating e^{iz} *along the boundary of the triangle with the vertices at the points* $1 + i, -1 + i$ *and* $-1 - i$.

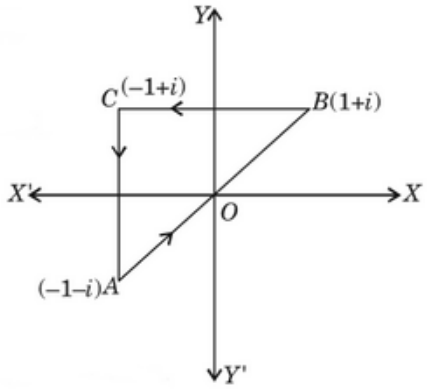
Solution

Let C be the boundary of the given triangle.

Given $f(z) = e^{iz}$

f(z) is analytic within and on closed contour C and therefore by Cauchy's theorem

$$\oint_C f(z)dz = 0$$



Again $\oint_C f(z)dz = \oint_{AB} f(z)dz + \oint_{BC} f(z)dz + \oint_{CA} f(z)dz$

Along AB

$y = x$

$\therefore \quad z = x + iy = x + ix = (1 + i)x$

$\therefore \quad dz = (1 + i)dx$

and x varies from -1 to 1.

$\therefore \quad \oint_{AB} f(z)dz = \oint_{AB} e^{i(1+i)x}(1+i)dx = \left[\frac{e^{(-1+i)x}(1+i)}{-1+i}\right]_{-1}^{1}$

$$= \frac{1+i}{i-1}\left[e^{-1+i} - e^{-(-1+i)}\right] = \frac{e^{-1+i} - e^{1-i}}{i}$$

Along BC

$y = 1$

$\therefore \quad z = x + iy = x + i$

$\therefore \quad dz = dx$

and x varies from 1 to -1.

$$\therefore \quad \oint_{BC} e^{iz}dz = \int_{1}^{-1} e^{i(x+i)}dx = e^{-1}\int_{1}^{-1} e^{ix}dx = e^{-1}\left[\frac{e^{ix}}{i}\right]_{1}^{-1} = \frac{e^{-1}}{i}(e^{-1} - e^{i}) = \frac{e^{-i-1} - e^{i-1}}{i}$$

Along CA

$x = -1$

$\therefore \quad z = x + iy = -1 + iy$

$\therefore \quad dz = idy$

and y varies from 1 to -1.

$$\therefore \quad \oint_{CA} f(z)dz = \oint_{CA} e^{iz}dz = \int_{1}^{-1} e^{i(-1+iy)}idy = \int_{1}^{-1} e^{-i-y}idy = e^{-1}i\int_{-1}^{1} e^{-y} = \frac{e^{-i+1} - e^{-i-1}}{i}$$

Now $$\oint_{C} f(z)dz = \oint_{AB} f(z)dz + \oint_{BC} f(z)dz + \oint_{CA} f(z)dz$$

$$= \frac{e^{-1+i} - e^{1-i}}{i} + \frac{e^{-i-1} - e^{i-1}}{i} + \frac{e^{-i+1} + e^{-i-1}}{i} = 0$$

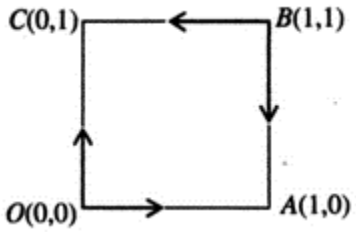
Example (JNTUK 2020, 7 marks)

Evaluate $\oint_C (z+1)dz$ *where C is the boundary of the square whose vertices are at the points z = 0, z = 1, z = 1 + i, z = i.*

Solution

See following figure.





Along OA

$y = 0$

$\therefore \quad z = x + iy = x$

$dz = dx$

$$\therefore \quad \oint_{OA} f(z)dz = \int_0^1 (x+1)dx = \left[\frac{x^2}{2} + x\right]_0^1 = \frac{3}{2}$$

Along AB

$x = 1 \Rightarrow z = 1 + iy \Rightarrow dz = idy$

$$\therefore \quad \oint_{AB} f(z)dz = i\int_0^1 (2+iy)dy = i\left[2y + i\frac{y^2}{2}\right]_0^1 = i\left[2 + \frac{i}{2}\right] = 2i - \frac{1}{2}$$

Along BC, $y = 1 \Rightarrow z = x + iy = x + i \Rightarrow dz = dx$

$$\therefore \quad \oint_{BC} f(z)dz = \int_1^0 (x+i+1)dx = \left[\frac{x^2}{2} = (i+1)x\right]_1^0 = -i - \frac{3}{2}$$

Along CO, $x = 0 \Rightarrow z = x + iy = iy \Rightarrow dz = idy$

$$\therefore \quad \oint_{CO} f(z)dz = i\int_1^0 (iy+1)dy = \left[i\frac{y^2}{2} + y\right] = -\frac{i}{2} - 1$$

Now $$\oint_C f(z)dz = \oint_{OA} + \oint_{AB} + \oint_{BC} + \oint_{CO} = \frac{3}{2} + 2i - i\frac{3}{2}\frac{i}{2} - 1 = \frac{1}{2}(i-3)$$

MAPPINGS

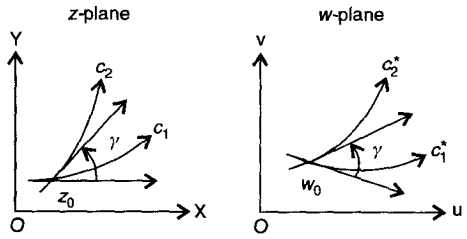
The set of equations

$$\left.\begin{aligned} u &= u(x,y) \\ v &= v(x,y) \end{aligned}\right\}$$

defines, in general, a transformation or mapping which establishes a correspondence between points in the uv and xy planes. The set of equations are called *transformation equations*.

Conformal Mapping

A mapping w = f(z) is said to be conformal if the angle between any two smooth curves c_1, c_2 in the z plane intersecting at the point z_0 is equal in magnitude and sense to the angle between their images c_1* and c_2* in the w plane at the point $w_0 = f(z_0)$. (See following figure).



Conformal mapping is used to map, complicated regions conformally onto simpler, standard regions such as circular disks, half planes and strips for which the boundary value problems are easy.

Bilinear or Fractional Transformation

The transformation

$$w = \frac{az+b}{cz+d}; ad - bc \neq 0$$

is called a bilinear or fractional transformation.

The transformation (3) has the property that circles in the z plane are mapped into circles in the w plane, where by circles we include circles of infinite radius which are straight lines.

The transformation maps any three distinct points of the z plane into three distinct points of the w plane, one of which may be infinity.

Example

Find the bilinear transformation which maps the points $z_1 = 2$, $z_2 = i$ and $z_3 = -2$ into the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$ respectively.

Solution

The bilinear transformation which transforms z_1, z_2 and z_3 respectively into w_1, w_2 and w_3 is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Substituting $z_1 = 2$, $z_2 = i$, $z_3 = -2$; $w_1 = i$, $w_2 = i$, $w_3 = -1$, we get

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)}=\frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

or $$-\frac{(w-1)(1+i)^2}{(w+1)(1-i)(1+i)}=-\frac{(z-2)(2+i)^2}{(z+2)(2-i)(2+i)}$$

or $$\frac{(w-1).2i}{(w+1).2}=\frac{(z-2)(3+4i)}{(z+2).5}$$

or $$\frac{w-1}{w+i}=\frac{(z-2)(4-3i)}{5(z+2)}$$

or $$\frac{2w}{2}=\frac{(z-2)(4-3i)+5(z+2)}{5(z+2)-(z-2)(4-3i)}$$

or $$w=\frac{3(3-i)z+2(1+3i)}{(1+3i)z+6(3-i)}=\frac{3z+[2(1+3i)/(3-i)]}{[(1+3i)/(3-i)]z+6}$$

or $w=\dfrac{3z+2i}{iz+6}$, which is the required transformation.

Example

Find transformation which maps outside $|z| = 1$ on the half plane $R(w) \geq 0$ so that the points $z = 1, -i, -1$ correspond to $w = i, 0, -i$ respectively.

Solution

Let $z_1 = 1$, $z_2 = -i$, $z_3 = -1$, $w_1 = i$, $w_2 = 0$, $w_3 = -i$.

Now $$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Putting values and proceeding

We get $$w=\frac{z+i}{z-i} \qquad (1)$$

This is the required transformation.

By (1), $$w(z-i)=z+i$$

or $$z=i\left|\frac{w+1}{w-1}\right| \qquad (2)$$



$|z| \geq 1$ is transformed into $\left|\frac{w+1}{w-1}\right|. |i| \geq 1$

or $|w+1|^2 \geq |w-1|^2$

or $(u+1)^2 + v^2 \geq (u-1)^2 + v^2$

or $R(w) = u \geq 0$

Thus exterior of the circle |z| = 1 is transformed into half plane $R(w) \geq 0$.

Problem

Find the bilinear transformation that maps the points $z_1 = \infty, z_2 = i$ *and* $z_3 = 0$ *into the points* $w_1 = 0, w_2 = i$ *and* $w_3 = \infty$.

Answer: $w = -\frac{1}{z}$



ASSIGNMENT

Q.1. (AKTU 2018, 2020, 2 marks): Define analytic function with an example.

Q.2. (UTU 2015, 5 marks): If f(z) is an analytic function with constant modulus, show that f(z) is a constant.

Q.3. (JNTUK 2021, 2 marks): Prove that f (z) = z^2 is analytic.

Answer: Solved in this module.

Q.4. (UTU 2013, 2018, 5 marks): Find the values of c_1 and c_2 such that the function

$f(z) = x^2 + c_1y^2 - 2xy + i(c_2x^2 - y^2 + 2xy)$

is analytic. Also find f'(z).

Answer: Solved in this module.

Q.5. (AKTU 2019, 2 marks): The function $f(x) = e^x(\cos y + i \sin y)$ is holomorphic or not.

Answer: Thus $u_x \neq v_y \; and \; u_y \neq -v_x$ for any x and y. CR conditions are not satisfied for any z. Hence $f(z) = e^{\bar{iz}}$ is nowhere analytic.

Q.6. (AKTU 2019, 2 marks): Let $f(z) = u(r, \theta) + iv(r, \theta)$ be an analytic function. If $u = -r^3\sin 3\theta$, then find v.

Answer: Similar problem is solved in this module.

Q.7. (AKTU 2021, 7 marks): Determine the analytic function f(z) = u + iv in terms of whose real part is e^x(xcosy - ysiny).

Answer: Solved in this module.

Q.8. (UTU 2015, 5 marks): Determine the analytic function whose real part is e^{2x}(xcos2y - ysin2y).

Answer: $f = e^{2x}[(x+iy)\cos 2y + i(x+iy)\sin 2y + ic]$

Q.9. (UTU 2016, 5 marks): Determine the analytic function whose real part is e^x(cosy + isiny).

Q.10. (JNTUK 2020, 7 marks): Determine analytic function whose real part is $u = e^{x^2-y^2}\cos 2y$.

Q.11. (PTU 2019, 5 marks): If $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ and f(z) = u + iv is an analytic function of z = x + iy, find f(z) in terms of z.

Answer: Solved in this module.

Q.12. (UTU 2016, JNTUK 2021, 7 marks): Find the analytic function f(z) = u + iv where v(x, y) = e^xcosy.

Q.13. (UTU 2017, 5 marks): If f(x) = u + iv is any analytic function of the complex variable z and v = e^{-x}(xcosy + ysiny), then find f(z) in terms of z.

Answer: f(z) = i(ze^{-z}) + c; use Milne Thompson method.

Q.14. (AKTU 2021, 2 marks): Show that the function f(z) = $x^2 - y^2$ is harmonic function.

Q.15. (AKTU 2018, 2 marks): Give an example of a function in which Cauchy Riemann Equations are satisfied yet the function is not analytic at the origin. Justify your answer.

Answer: $|xy|^{1/2}$. Solved in detail in this module.



Q.16. (UTU 2015, JNTUK 2021, 7 marks): Show that $f(z) = \sqrt{|xy|}$ is not analytic at z = 0 although the CR equations are satisfied at the origin.

Answer: Solved in this module.

Q.17. (UTU 2013, 2017, 5 marks): Show that the function f(z) is defined by

$f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}}, z \neq 0, f(0) = 0$ is not analytic at the origin even though it satisfies Cauchy-Riemann equation at the origin.

Answer: Solved in this module.

Q.18. (AKTU 2019, 3.5 marks): Expand $\frac{1}{z^2 - 3z + 2}$ in the region 1 , |z| < 2.

Answer: Solved in this module.

Q.19. (PTU 2019, 2 marks): State Cauchy's integral formula.

Q.20. (PTU 2018, 5 marks): State and prove the Cauchy's integral formula.

Q.21. (PTU 2019, 2 marks): Evaluate $\int \frac{e^z}{(z+1)^2} dz$ along the circle |z - 3| = 3.

Q.22. (JNTUK 2020, 2 marks): Evaluate $\oint_C \frac{e^z}{z-1} dz$ where C is the circle |z| = 1/2.

Q.23. (PTU 2018, 2 marks): Evaluate $\int \frac{z^3}{z+i} dz$ along the circle |z| = 2.

Q.24. (AKTU 2020, 7 marks): State Cauchy integral formula and hence evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle |z| = 3.

Answer: Solved in this module.

Q.25. (AKTU 2021, 7 marks): State and prove the Cauchy integral formula. Also evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)} dz$ where C is circle |z| = 3/2.

Answer: Solved in this module.

Q.26. (JNTUK 2021, 7 marks): Evaluate $\int_C \frac{ze^z}{(z-a)^3} dz$ where "a" lies within a closed curve by Cauchy integral formula.

Answer: Solved in this module.

Q.27. (UTU 2016, 5 marks): Evaluate by Cauchy's integral formula $\oint_C \frac{dz}{z(z+\pi i)}$



Q.28. (UTU 2017, 5 marks): Using Cauchy integral formula evaluate the following integral

$f_c \frac{e^{2z}dz}{z-\pi i}$, where C is the ellipse |z - 2| + |z + 2| = 6.

Q.29. (AKTU 2019, 3.5 marks): Verify Cauchy theorem by integrating e^{iz} along the boundary of the triangle with the vertices at the points $1 + i, -1 + i \; and - 1 - i$.

Answer: Solved in this module.

Q.30. (JNTUK 2020, 7 marks): Evaluate $\oint_C (z+1)dz$ where C is the boundary of the square whose vertices are at the points z = 0, z = 1, z = 1 + i, z = i.

Answer: Solved in this module.

Q.31. (UTU 2018, 5 marks): Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along the line 2y = x.

Q.32. (PTU 2019, 2 marks): Find the bilinear transformation that map the points z = 0, -i, -1 into the points w = i, 1, 0.

Answer: $w = i\left(\frac{1+z}{1-z}\right)$

Q.33. (JNTUH 2012, 2013, 2014, UTU 2017, 5 marks): If f(z) is regular function of z, show that

$$\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}|f(z)|^2 = 4|f'(z)|^2$$

Answer: Solved in this module.

Q.34. (JNTUK 2004, 2005, 2020, 7 marks): If f(z) is an analytic function show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^2 = 2|f'(z)|^2$$

Answer: Solved in this module.

