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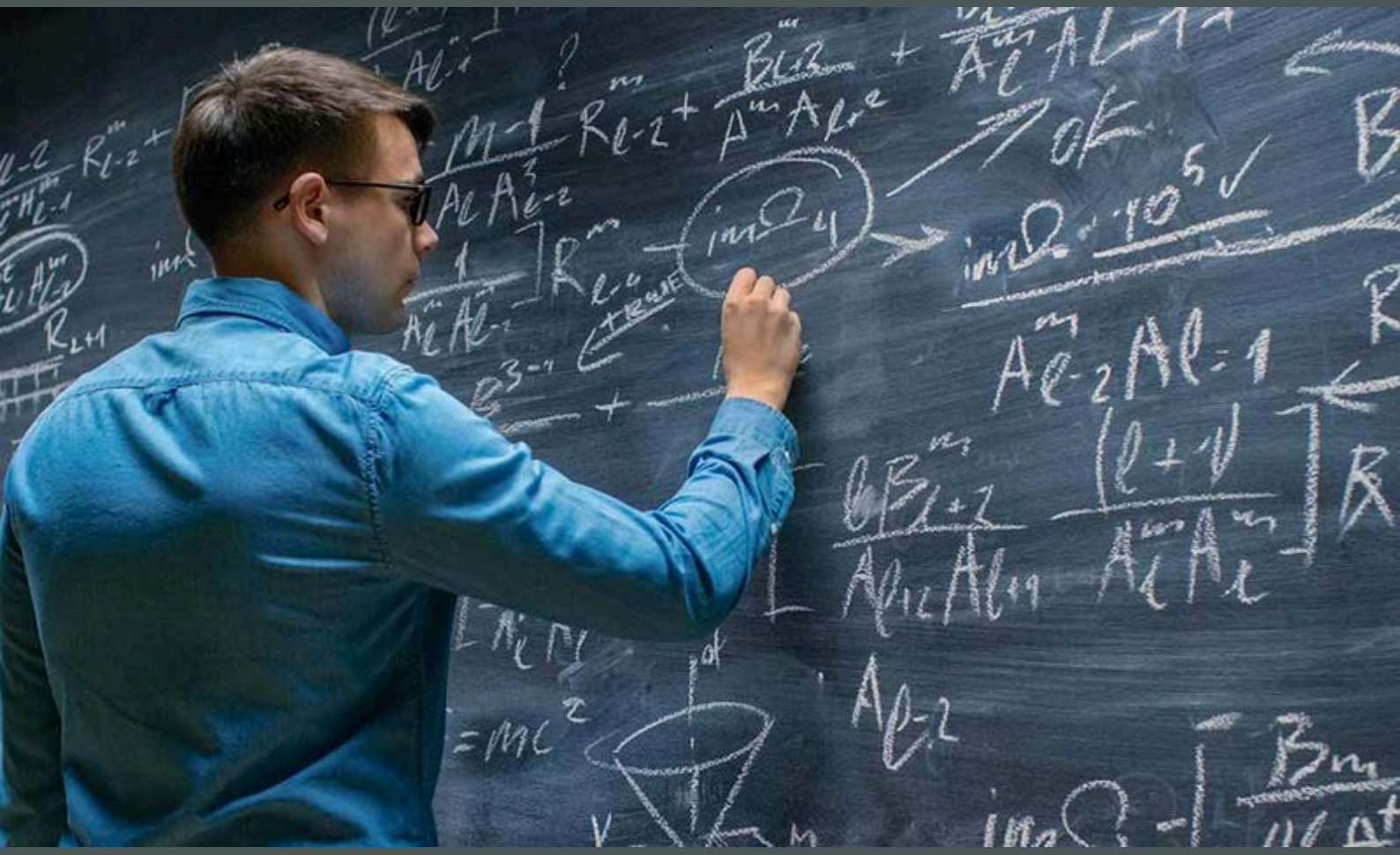

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NUMERICAL METHODS - II

NUMERICAL DIFFERENTIATION

In next new sections, we shall be concerned with the problems of numerical differentiation and integration. That is to say, given the set of values of x and y , we shall write formulae to compute

- $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, ……for any value of x in $[x_0 , x_n]$, and
- $\int_{x_0}^{x_n} y dx$

If a derivative is required **near the beginning** of a table, formulae may be used to obtain better accuracy:

$$h.\frac{dy}{dx} = hy'_0 = \left(\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \frac{1}{5}\Delta^5 - \frac{1}{6}\Delta^6 + ...\right)y_0 \quad (1)$$

and
$$h^2 y''_0 = \left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \frac{137}{180}\Delta^6 - \frac{7}{10}\Delta^7 + \frac{363}{560}\Delta^8 + ...\right)y_0 \quad (2)$$

If derivative is required **near and of a table**

$$hy'_n = \left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \frac{1}{5}\nabla^5 + \frac{1}{6}\nabla^6 + \frac{1}{7}\nabla^7 + \frac{1}{8}\nabla^8 + ...\right) \quad (3)$$

and
$$h^2 y''_0 = \left(\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \frac{137}{180}\nabla^6 + \frac{7}{10}\nabla^7 + \frac{363}{560}\nabla^8 + ...\right)y_n \quad (4)$$

Example

From the following table of values of x and y, obtain $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *for x = 1.2*

x	*1.0*	*1.2*	*1.4*	*1.6*	*1.8*	*2.0*	*2.2*
y	*2.7183*	*3.3201*	*4.0552*	*4.9530*	*6.0496*	*7.3891*	*9.0250*

Solution

The difference table is



x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

Here $x_o = 1.2$, $y_o = 3.3201$ and $h = 0.2$.

Hence (1) gives

$$\left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014\right]$$

$$= 3.3205$$

Similarly, formula (2) gives :

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{0.04}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right] = 3.318$$

Example

Calculated the first and second derivatives of the function tabulated in the preceding Example at the point x = 2.2

Solution

We use table of difference of above example.

Here $x_n = 2.2$, $y_n = 9.0250$ and $h = 0.2$.

Hence formula (3) gives :



$$\left[\frac{dy}{dx}\right]_{x=2.2} = \frac{1}{0.2}\left[1.6359 + \frac{1}{2}(0.2964) + \frac{1}{3}(0.0535) + \frac{1}{4}(0.0094) + \frac{1}{5}(0.0014\right]$$

$$= 9.0228$$

Formula (4) gives

$$\left[\frac{d^2y}{dx^2}\right]_{x=2.2} = \frac{1}{0.04}\left[0.2964 - 0.0535 + \frac{11}{12}(0.0094) + \frac{5}{6}(0.0014)\right] = 8.992$$

Problem

Given that

x	*1.0*	*1.1*	*1.2*	*1.3*	*1.4*	*1.5*	*1.6*
y	*7.989*	*8.403*	*8.781*	*9.129*	*9.451*	*9.750*	*10.031*

find dy/dx and d^2y/dx^2 at (a) x = 1.1 (b) x = 1.6

Answer: (a) 3.946, -3.545 (b) 2.727, -1.703

Example (AKTU 2018, 2020, 2021, 7 marks)

Find x for which y is maximum and find the max value of y

x	*1.2*	*1.3*	*1.4*	*1.5*	*1.6*
y	*0.9320*	*0.9636*	*0.9855*	*0.9975*	*0.9996*

Solution

Let us prepare the following table:

x	x = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 1.2$	$y_0 = 0.9320$				
		$\Delta y_0 = 0.0316$			
$x_1 = 1.3$	$y_1 = 0.9636$		$\Delta^2 y_0 = -0.0097$		
		0.0219		$\Delta^3 y_0 = -0.0002$	
$x_2 = 1.4$	$y_2 = 0.9855$		-0.0099		$\Delta^4 y_0 = 0.0002$
		0.0220		*0*	
$x_3 = 1.5$	$y_3 = 0.9975$		*-0.0099*		
		∇y_4 = 0.0021			
$x_4 = 1.6$	$y_4 = 0.9996$				



The value of x correct upto two digits is required. From the above table it is clear that if we consider upto second difference ($\Delta^2 y$), x will be correct upto two digits. Putting these values in following equation

$$0.0316 + \frac{2p-1}{2}(-0.0097) = 0 \text{ (neglecting 3rd and higher order differences)}$$

From this, p = 3.757732

We know $x_p = x_0 + ph$

where $h = x_1 - x_0 = 1.3 - 1.2 = 0.1$ and $x_0 = 1.2$.

Putting these values

$$x_p = 1.2 + (3.757732)(0.1) = 1.5757732$$

Thus, "y" is maximum at $x_p = 1.5757732$.

We know that y is maximum at $x_p = 1.5757732$ observe that this value of x is near *end* of data points. Hence 'y' can be obtained using **newton's backward differences** interpolation. We know that,

$$p = \frac{x_p - x_n}{h} \text{ for backward differences.}$$

Here $h = 0.1$, $x_n = x_4 = 1.6$ and $x_r = 1.5757732$

$$\therefore \quad p = \frac{1.5757732 - 1.6}{0.1} = -0.242268$$

Newton's backward differences interpolation formula is given as

$$y_r = y_n + r\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + ..$$

Putting values,

$$y_p = 0.9996 + (-0.242268)(0.0021) + \frac{(-0.242268)(-0.242268+1)}{2!}(-0.0099) + 0$$
$$+ \frac{(-0.242268)(-0.242268+1)(-0.242268+2)(-0.242268+3)}{4!}(0.0002)$$

$$= 0.999993$$

Example (KU 2019, 7.5 marks)

From the following data, find the maximum or minimum value of y.

x	*0.60*	*0.65*	*0.70*	*0.75*



y	*0.6221*	*0.6155*	*0.6138*	*0.6170*

Solution

Forward difference table for given data is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.60	0.6221			
		-0.0066		
0.65	0.6155		0.0049	
		-0.0017		0
0.70	0.6138		9	
		0.032		
0.75	0			

$x_0 = 0.60$, $h = 0.05$, $p = (x - x_0)/h = (x - 0.60)/0.05 = 20x - 12$

By Newton's forward difference interpolation formula

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \ldots$$

$$\therefore \quad y_p = 0.6221 - 0.0066p + \frac{0.0049}{2}(p^2 - p)$$

$$\frac{dy}{dp} = -0.0066 + \frac{0.0049}{2}(2p - 1)$$

For maxima or minima

$$\frac{dy}{dx} = \frac{dy}{dp}.\frac{dp}{dx} = \frac{dy}{dp}(20) = 0$$

$$\therefore \quad -0.0066 + \frac{0.0049}{2}(2p - 1) = 0$$

which gives $p = 1.85$

Now $$\frac{d^2y}{dx^2} = 20.\frac{d^2y}{dp^2}.\frac{dp}{dx} = 400\frac{d^2y}{dp^2} = 400(0.0049) > 0$$

$\therefore$ y is minimum when $p = 1.85$.

$$\therefore \quad y_{\min} = 0.6221 - 0.0066(1.85) + \frac{0.0049}{2}(1.85)(0.85)$$



= 0.6137

NUMERICAL INTEGRATION

The general problem of numerical integration may be stated as follows. Given a set of data points (x_0,y_0), (x_1,y_1),……(x_n,y_n) of a function y = f(x) , where f(x) is not known explicitly, it is required to compute the value of the definite integral

$$I = \int_a^b y\, dx \qquad (10)$$

As in the case of numerical differentiation, one replaces f(x) by an interpolating polynomial $\phi(x)$ and obtains, on integration, an *approximate value* of the definite integral. Thus, different integration formula can be obtained depending upon the type of the interpolation formula used. Following is the general formula for numerical integration using Newton's forward difference formula.

$$\int_{x_0}^{x_n} y\, dx = n\,h\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}.\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + ... \right] \qquad (11)$$

from this *general formula*, we can obtain different integration formula by putting n = 1,2,3,…, etc. We derive here a few of these formula but it should be remarked that the trapezoidal and Simpson's $\frac{1}{3}$ rules are found to give sufficient accuracy for use in practical problems.

Trapezoidal Rule

Setting n = 1 in the general formula (11), and going through a few calculations

Combining all these expression, we obtain the rule

$$\int_{x_0}^{x_n} y\, dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n] \qquad (12)$$

which is known as the ***Trapezoidal rule***.

The geometrical significance of this rule is that the curve y = f(x) is replaced by n straight lines joining the points (x_0,y_0) and (x_1,y_1) and (x_2,y_2);……;(x_{n-1},y_{n-1}) and (x_n,y_n).

The area bounded by the curve y = f(x), the ordinates $x = x_0$ and $x = x_n$, and the x-axis is then approximately equivalent to the sum of the areas of the n trapeziums obtained.

Simpson's 1/3 Rule

This rule is obtained by putting n = 2 in eq…(11)



$$\int_{x_2}^{x_n} y\,dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \ldots\ldots + y_{n-2}) + y_n] \quad (13)$$

Which is known as **Simpson's 1/3-rule, or simply Simpson's rule**.

It should be noted that this rule requires the division of the whole range into an even number of subinterval of width h.

Example

Find, from the following table, the area bounded by the curve and the x-axis from x = 7.47 to x = 7.52

x	*7.47*	*7.48*	*7.49*	*7.50*	*7.51*	*7.52*
f(x)	*1.93*	*1.95*	*1.98*	*2.01*	*2.03*	*2.06*

Solution

We know that

$$\text{Area} = \int_{7.47}^{7.52} f(x)\,dx$$

With h = 0.01, the trapezoidal rule (12) gives

$$\text{Area} = \frac{0.01}{2}.[1.93 + 2(1.95 + 1.98 + 2.01 + 2.03) + 2.06] = 0.0996$$

Example

A solid of revolution is formed by rotating about the x-axis the area between the x-axis, the lines x = 0 and x = 1, and a curve through the points with the following coordinates

x	*0.00*	*0.25*	*0.50*	*0.75*	*1.00*
y^2	*1.0000*	*0.9793*	*0.9195*	*0.8261*	*0.7081*

Estimate the volume of the solid formed, giving the answer to three decimal places.

Solution

If V is the volume of the solid formed, then we know that

$$V = \pi \int_0^1 y^2\,dx$$

Hence we need the values of y^2 and these are tabulated below, correct to four decimal places



x	0.00	0.25	0.50	0.75	1.00
y^2	1.0000	0.9793	0.9195	0.8261	0.7081

With h = 0.25, Simpson's rule gives

$$V = \frac{\pi(0.25)}{3}.[1.0000 + 4(0.9793) + 0.8261) + 2(0.9195) + 0.7081]$$

$$= 2.8192$$

Example

Evaluate $I = \int_0^1 \frac{1}{1+x}\,dx$, *correct to three decimal places*

Solution

We solve this example by both the trapezoidal and Simpson's rules with h = 0.5, 0.25 and 0.125 respectively.

(i) h = 0.5. The values of x and y are tabulated below

x	0	0.5	1.0
y	1.0000	0.6667	0.5

(a) Trapezoidal rule gives

$$I = \frac{1}{4}[1.0000 + 2(0.6667) + 0.5] = 0.7084$$

(b) Simpson's rule gives

$$I = \frac{1}{6}[1.0000 + 4(0.6667) + 0.5] = 0.6945$$

(ii) h = 0.25. The tabulated values of x and y are given below

x	0	0.25	0.50	0.75	1.00
y	1.0000	0.8000	0.6667	0.5714	0.5

(a) Trapezoidal rule gives

$$I = \frac{1}{8}[1.0 + 2(0.8000 + 0.6667 + 0.5714) + 0.5] = 0.6970$$

(b) Simpson's rule gives



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$$I = \frac{1}{12}[1.0 + 4(0.8000 + 0.5714) + 2(0.6667) + 0.5] = 0.6932$$

(iii) Finally, we take h = 0.125. The tabulated values of x and y are

x	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1.0
y	1.0	0.8889	0.8000	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

(a) Trapezoidal rule gives

$$I = \frac{1}{16}[1.0 + 2(0.8889 + 0.8000 + 0.7273) + 0.6667 + 0.6154 + 0.5714 + 0.5333 + 0.5]$$
$$= 0.6941$$

(b) Simpson's rule gives

$$I = \frac{1}{24}[1.0 + 4(0.8889 + 0.7273 + 0.6154 + 0.5333 + 2(0.8000 + 0.6667 + 0.5714 + 0.5]$$
$$= 0.6932$$

Hence the value of I may be taken to be equal to 0.693, correct to three decimal places. The exact value of I is $\log_e 2$, which is equal to 0.693147…..

This example demonstrates that, in general, Simpson's rule yields more accurate results than the trapezoidal rule.

Simpson's 3/8 Rule

This rule is

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}[(y_0 + 3y_1 + 2y_3 + 3y_4 + 3y_5 + 2y_0 + \ldots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

$$= \frac{3h}{8}\left[y_0 + 3(y_1 + y_2 + y_4 + \ldots y_{n-1}) + 2(y_3 + y_0 + \ldots . y_{n-3}) + y_n\right]$$

This rule is not so accurate as Simpson's rule.

Example (AKTU 2018, 2019, 7 marks)

Evaluate $\int_0^{\pi/2} e^{\sin x} dx$ *correct to four decimal places by Simpson's one-third and three-eight rule, dividing the interval (0, π/2) into three equal parts.*



Solution

Divide the interval (0, π/2) into three parts each of width h = [(π/2) - 0]/3 = π/6 and compute the value of $y = e^{\sin x}$ at each point of sub interval. These values are given below:

x	0	π/6	π/3	π/2
$y = e^{\sin x}$	1	1.64872	2.45960	2.71828

Simpson's one-third rule

$$\int_0^{\pi/2} e^{\sin x} dx = \frac{h}{3}[y_0 + 4y_1 + 2y_3]$$

$$= \frac{\pi/6}{3}[1 + 4x1.64872 + 2x2.45960 + 2.71828] = 2.6596$$

Simpson's 3/8 rule

$$\int_0^{\pi/2} e^{\sin x} dx = \frac{3h}{8}[y_0 3(y_1 + y_2) + y_3]$$

$$= \frac{3}{8}.\frac{\pi}{6}[1 + 3(1.64872 + 2.459600 + 2.71828] = 3.1513$$

Example (AKTU 2019, UTU 2015, 2017, 10 marks)

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ *by using Trapezoidal, Simpson's one-third and three-eighth rule.*

Solution

Divide the interval (0,6) into six parts each of width h = 1 and compute the value of $y = 1/(1 + x^2)$ at each point of sub-interval. These values are given below

x	$y = \frac{1}{1+x^2}$
0	1
1	0.5
2	0.2
3	0.1
4	0.0588
5	0.0385



6	0.027

Trapezoidal rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6\right]$$

$$= \frac{1}{2}\left[1 + 2(0.5 = 0.2 + 0.1 + 0.0588 + 0.0385) + 0.027\right] = 1.4108$$

Simpson's 1/3 rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + y_6\right]$$

$$= \frac{1}{3}\left[1 + 4(0.5 + 0.1 = 0.0385) + 2(0.2 + 0.0588) + 0.027\right] = 1.3662$$

Simpson's 3/8 rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[y_0 + 3\left(y_1 + y_2 + y_4 + y_5\right) + 2\left(y_3\right) + y_6\right]$$

$$= \frac{3}{8}\left[1 + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1) + 0.027\right] = 1.3570$$

Problem

Calculate the approximate value of $\int_{-3}^{3} x^4 dx$ *by using Trapezoidal rule, Simpson's 1/3 and 3/8 rules by dividing the range in six equal parts.*

Answer: 115, 98, 99

Numerical Solution Of ODE

Many problems in science and engineering can be reduced to the problem of solving differential equations satisfying certain given conditions. The analytical methods of solution, with which the reader is assumed to be familiar, can be applied to solve only a selected class of differential equations. Those equations which govern physical systems do not posses, in general, closed form solutions, and hence recourse must be made to numerical methods for solving such differential equations.



SOLUTION BY TAYLOR'S SERIES

We consider the differential equation

$$\frac{dy}{dx} = y' = f(x, y) \quad initial\ condition\ y(x_0) = y_0 \qquad (1)$$

If y(x) is the exact solution of (1), then the Taylor's series for y(x) around x = x_0 is given by

$$y(x) = y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \ldots\ldots\ldots\ldots \qquad (2)$$

Example

From the Taylor's series for y(x), find y(0.1) correct to four decimal places if y(x) satisfies

$$y' = x - y^2 \quad and \quad y(0) = 1$$

Solution

The Taylor series for y(x) is given by

$$y(x) = 1 + x\, y_0' + \frac{x^2}{2} y_0'' + \frac{x^3}{6} y_0''' + \frac{x^4}{24} y_0^{iv} + \frac{x^5}{120} y_0^{v} + \ldots\ldots\ldots\ldots$$

The derivatives $y_0'', y_0'', \ldots\ldots$ etc. are obtained thus:

$$y'(x) = x - y^2 \qquad y_0' = -1$$

$$y''(x) = 1 - 2yy' \qquad y_0'' = 3$$

$$y'''(x) = -2yy'' - 2y'^2 \qquad y_0''' = -8$$

$$y^{iv}(x) = -2yy''' - 6y'y'' \qquad y_0^{iv} = 34$$

$$y^{v}(x) = -2yy^{iv} - 8y'y''' - 6y''^2 \qquad y_0^{v} = -186$$

Using these values, the Taylor series becomes

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 - \frac{17}{12}x^4 - \frac{31}{20}x^5 + \ldots\ldots$$

To obtain the values of y(0.1) correct to four decimal places, it is found that the terms upto x^4 should be considered, and we have y(0.1) = 0.9138.

Problem

Using Taylor-series method solve



$\frac{dy}{dx} = x^2 - y$, $y(0) = 1$ *at* $x = 0.1$.

Answer: 0.9052,

EULER'S METHOD

The general formula is given below

$$y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0, 1, 2, ...$$

The process is *very slow* and to obtain reasonable accuracy with Euler's method, we need to take a smaller value for h. Because of this restriction on h, the method is unsuitable for practical use and a modification of it, known as the *modified Euler method*, which gives more accurate results, will be described in next section.

Example

Solve for y the differential equation $\frac{dy}{dx} = x + y, y(0) = 1$ *at x = 0.0, 0.2 and 0.8 using Euler's method.*

Solution

Given $\frac{dy}{dx} = f(x, y) = x + y = x + y; y(0) = 1, h = 0.2$

n	x_n	y_n	$f(x_n,y_n)$	$y_{n+1} = y_n + hf(x_n, y_n)$
0	0.0	1.0000	1.0000	1.0000 + 0.2(1.0000) = 1.2000
1	0.2	1.2000	1.4000	1.2000 + 0.2(1.4000) = 1.4800
2	0.4	1.4800	1.8800	1.8560
3	0.6	1.8560	2.4560	2.3472
4	0.8	2.3472		

$\therefore$ y(0.2) = 1.2000, y(0.4) = 1.4800, y(0.6) = 1.8560, y(0.8) = 2.3472

Example

Use Euler's method compute y(0.5) for differential equation $\frac{dy}{dx} = y^2 - x^2$ *with y = 1 when x = 0.*



Solution

We have $\frac{dy}{dx} = y^2 - x^2 = f(x, y), x_0 = 0, y_0 = 1$

Let h = 0.5/5 = 0.1

Using the formula

$$y_{n+1} = y_n + hf(x_n, y_n)$$

By putting n = 0, 1, 2, 3, 4, we get

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(y_0^2 - x_0^2) = 1 + (0.1)(1^2 - 0^2) = 1.1$$

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + h(y_1^2 - x_1^2)$$

$= 1.1 + (0.1)[(1.1)^2 - (0.1)^2] = 1.220$ [$[x_1 = x_0 + h = 0 + 0.1 = 0.1]$

$$y_3 = y_2 + hf(x_2, y_2) = y_2 + h(y_2^2 - x_2^2)$$

$= 1.22 + (0.1)[(1.22)^2 - (0.2)^2]$ $[x_2 = x_1 + h = 0.1 + 0.1 = 0.2]$

$= 1.36484$

Similarly $y_4 = 1.54212$; $y_5 = 1.7639$

We can stop here, y(0.5) = 1.7639.

If we proceed further, we may get more precise value of y(0.5).

Problem

Use Euler's method compute y(0.04) for the differential equation $\frac{dy}{dx} = -y$ *with y = 1 at x = 0.*

Example

Solve $\frac{dy}{dx} = y - \frac{2x}{y}$, *y(0) = 1 in the range* $0 \le x \le 0.2$ *by Euler's method.*

Solution

$$\frac{dy}{dx} = f(x, y) = y - \frac{2x}{y}; y(0) = 1, h = 0.1$$

x_n	y_n	$f(x_n,y_n)$	$y_{n+1} = y_n + hf(x_n, y_n)$
0	1	1	1.1
0.1	1.1	0.9182	1.1918
0.2	1.1918		



$\therefore$ y(0.1) = 1.1, y(0.2) = 1.192

Problem

Solve $\frac{dy}{dx} = 1 - y, y(0) = 0$ *in the range* $0 \le x \le 0.3$ *using Euler's method.*

Answer: y (0.1) = 0.1, y (0.2) = 0.19, y (0.3) = 0.271

Modified Euler's Method

Instead of approximating f(x, y) by $f(x_0,y_0)$, we now approximate the integral by means of trapezoidal rule to obtain

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] \qquad (1)$$

We thus obtain the iteration formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right] \quad n = 0,1,2,..... \qquad (2)$$

where $y_1^{(n)}$ is the nth approximation to y_1. The Euler's iteration formula $y_{n+1} = y_n + hf(x_n, y_n)$ can be started by choosing $y_1^{(0)}$ from Euler's formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

We repeat the steps till the two values of y become same to the desired accuracy.

Example

Determine the value of y when x = 0.1 given that y(0) = 1 and y' = $x^2 + y$.

Solution

We take h = 0.05. With $x_0 = 0$ and $y_0 = 1.0$, we have $f(x_0,y_0) = 1.0$.

Hence Euler's formula gives

$$y_1^{(0)} = 1 + 0.05(1) = 1.05$$

Further $x_1 = 0.05$ and $f(x_1, y_1^{(0)}) = 1.0525$. The average of $f(x_0,y_0)$ and $f(x_1, y_1^{(0)})$ is 1.0262. The value of $y_1^{(1)}$ can therefore be computed by using (vii) and we obtain

$$y_1^{(1)} = 1.0513$$

Repeating the procedure, we obtain $y_1^{(2)} = 1.0513$.



Hence we take $y_1 = 1.0513$, which is correct to four decimal places.

Next, with $x_1 = 0.05$, $y_1 = 1.0513$ and $h = 0.05$, we continue the procedure to obtain y_2, i.e., the value of y when x = 0.1. The results are

$$y_2^{(0)} = 1.1040, \quad y_2^{(1)} = 1.1055 \quad and \quad y_2^{(2)} = 1.1055$$

Hence we conclude that the value of y when x = 0.1 is 1.1055.

Example (BPUT 2020, 16 marks)

Using Euler's modified method, compute y(0.1) correct to six decimal figures where $\frac{dy}{dx} = x^2 + y$ *with y(0) = 0.94.*

Solution

Here $f(x, y) = x^2 + y, x_0 = 0, y_0 = 0.94, h = 0.1$

By Euler's method

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h[x_0^2 + y_0]$$

$$= 0.94 + (0.1)[0 + 0.94] = 1.034$$

The value of y_1, thus obtained is improved by Euler's modified method

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)}) \right]$$

Putting n = 0 $\quad y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$

$$= 0.94 + \frac{0.1}{2}\left[(0 + 0.94) + [(0.1)^2 + 1.034) \right] = 1.0392$$

Put n = 1 $\quad y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]$

$$= 0.94 + \frac{0.1}{2}\left[(0 + 0.94) + [(0.1)^2 + (1.0392)] \right] = 1.03946$$

Similarly

Put n = 2 $\quad y_1^{(3)} = 1.039473$

Put n = 3 $\quad y_1^{(4)} = 1.039473$

So, we see that $y_1^{(3)} = y_1^{(4)}$



$\therefore$ $y(0.1) = 1.039473$

Problem

Find y(2.2) for $\frac{dy}{dx} = -xy^2$ *where y(2) = 1 by Euler's modified method.*

Answer: 0.7018

Problem

Given $\frac{dy}{dx} = x^2 + y, \quad y(0) = 1$, *determine y(0.02), y(0.04) and (0.06) using Euler's modified method.*

Answer: 1.0202, 1.0408, 1.0619

PICARD'S METHOD

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx$$

with $y^{(0)} = y_0$

We repeat the steps till the two values of y become same to the desired accuracy.

Example (AKTU 2019, 3.5 marks)

Use Picard's method to solve $\frac{dy}{dx} x - y$ *for x = 0.1 and 0.2 given that y = 1 when x = 0.*

Solution

We have $f(x, y) = x - y$ and $x_0 = 0$, $y_0 = 1$

First approximation

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx = y_0 + \int_{x_0}^{x} (x - y_0)dx$$

$$= 1 + \int_0^x (x-1)dx = 1 + \left[\frac{x^2}{2} - x\right]_0^x = \frac{x^2}{2} - x + 1$$

Second approximation

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx = y_0 + \int_{x_0}^{x} (x - y^{(1)})dx$$



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$$= 1+\int_0^x \left[x-\left(\frac{x^2}{2}-x+1\right)dx \right] = 1+\int_0^x \left(2x-\frac{x^2}{2}-1\right)dx$$

$$= 1+\left[x^2-\frac{x^3}{6}-x \right]_0^x = -\frac{x^3}{6}+x^2-x+1$$

Third approximation

$$y^{(3)} = y_0+\int_{x_0}^x f(x,y^{(2)})dx = y_0+\int_{x_0}^x (x-y^{(2)})dx$$

$$=1+\int_0^x \left(x+\frac{x^3}{6}-x^2+x-1 \right)dx = 1+\left(x^2+\frac{x^4}{24}-\frac{x^3}{3}-x \right)_0^x$$

$$= \frac{x^4}{24}-\frac{x^3}{3}+x^2-x+1$$

Fourth approximation

$$y^{(4)} = y_0+\int_{x_0}^x f(x,y^{(3)})dx = y_0+\int_{x_0}^x (x-y^{(3)})dx$$

$$= 1+\int_0^x \left[x-\left(\frac{x^4}{24}-\frac{x^3}{3}+x^2-x+1\right) \right]dx$$

$$= \frac{x^5}{120}+\frac{x^4}{12}-\frac{x^3}{3}+x^2-x+1$$

Fifth approximation

$$y^{(5)} = y_0+\int_{x_0}^x f(x,y^{(4)})dx = y_0+\int_{x_0}^x (x-y^{(4)})dx$$

$$= 1+\int_0^x \left[x-\left(-\frac{x^5}{120}+\frac{x^4}{12}-\frac{x^3}{3}+x^2-x+1\right) \right]dx$$

$$= \frac{x^6}{720}-\frac{x^5}{60}+\frac{x^4}{12}-\frac{x^3}{3}+x^2-x+1$$

When x = 0.1, we have

$y^0 = 1$, $y^{(1)} = 0.905$, $y^{(2)} = 0.9098$, $y^{(3)} = 0.90967$, $y^{(4)} = 0.90967$

i.e., $y = 0.90967$ at $x = 0.1$

When x = 0.2, we have

$y^0 = 1$, $y^{(1)} = 0.82$, $y^{(2)} = 0.83867$, $y^{(3)} = 0.83740$, $y^{(4)} = 0.83746$



$y^{(5)} = 0.83746$

i.e., $y = 0.83746$ at $x = 0.2$.

Example

Apply Picard's method to solve $\frac{dy}{dx} = x + y^2$ *given that when* $x_0 = 0$, $y_0 = 0$ *upto third order of approximation.*

Solution

We have $f(x, y) = x + y^2$ and $x_0 = 0$; $y_0 = 0$

First approximation

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx = y_0 + \int_{x_0}^{x} (x + y_0^2)dx = 0 + \int_0^x (x + 0)dx$$

$$= \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$

Second approximation

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx = y_0 + \int_{x_0}^{x} [x + (y^{(1)})^2]dx$$

$$= y^{(2)} = 0 + \int_0^x \left[x + \left(\frac{x^2}{2}\right)^2\right]dx = \int_0^x \left(x + \frac{x^4}{4}\right)dx = \frac{x^2}{2} + \frac{x^5}{20}$$

Third approximation

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})dx = y_0 + \int_{x_0}^{x} [x = (y^{(2)})^2]dx$$

$$= 0 + \int_0^x \left[x + \left(\frac{x^2}{2} + \frac{x^5}{20}\right)^2\right]dx = \int_0^x \left(x + \frac{x^4}{4} + \frac{x^{10}}{400} + \frac{x^7}{20}\right)dx$$

$$= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}$$

Problem

Approximate y and z by using Picard's method for the particular solution of

$$\frac{dy}{dx} = x + z$$



and $\frac{dz}{dx} = x - y^2$

for x = 0.1 given that y = 2, z = 1 where x = 0.

Solution

Following table gives approximations.

n	$y^{(n)}(x) = 2 + \frac{x^2}{2} + \int_0^x z^{(n-1)}(x)dx$	$z^{(n)}(x) = 1 + \frac{x^2}{2} - \int_0^x [y^{(n-1)}]^2(x)dx$
0	2	1
1	$2 + x + \frac{x^2}{2}$	$1 + \frac{x^2}{2} - \int_0^x 4dx$ $= 1 - 4x + \frac{x^2}{2}$
2	$2 + x - \frac{3}{2}x^2 + \frac{x^3}{6}$	$1 + \frac{x^2}{2} - \int_0^x (4 + 4x + 3x^2 + x^3)dx$ $= 1 - 4x - \frac{3}{2}x^2 - x^3 - \frac{x^4}{4}$
3	$2 + x - \frac{3}{2}x^2 - \frac{x^3}{2} - \frac{x^4}{4}$	$1 + \frac{x^2}{2} - x\int_0 \left(4 + 4x - 5x^2 - \frac{7}{3}x^3\right)dx$ $= 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{12}x^4$
4	$2 + x - \frac{3}{2}x^2 - \frac{x^3}{2} + \frac{5}{12}x^4$	$1 + \frac{x^2}{2} - \int_0^x (4 + 4x - 5x^2 - 5x^3)dx$ $= 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{5}{4}x^4$
5	$2 + x - \frac{3}{2}x^2 - \frac{x^3}{2} + \frac{5}{12}x^4$	$1 + \frac{x^2}{2} - \int_0^x (4 + 4x - 5x^2 - 5x^3)dx$ $= 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{5}{4}x^4$

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From last approximations of y and z

$$y(x) = 2 + x - \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{5}{12}x^4$$

$$z(x) = 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{5}{4}x^4$$



$$\therefore \qquad y(0.1) = 2 + (0.1) - \frac{3}{2}(0.1)^2 - \frac{1}{2}(0.1)^3 + \frac{5}{12}(0.1)^4 = 2.0845$$

and $$z(0.1) = 1 - 4(0.1) - \frac{3}{2}(0.1)^2 + \frac{5}{3}(0.1)^3 + \frac{5}{4}(0.1)^4 = 0.5868$$

Problem

Use Picard's method to approximate the value of y when x = 0.1 given that y = 1 when x = 0

$$\frac{dy}{dx} = 3x + y^2$$

Answer: y(0.1) = 1.127

RUNGE-KUTTA METHOD

As already mentioned, Euler's method is less efficient in practical problems since it requires h to be small for obtaining reasonable accuracy. The *Runge-Kutta methods* are designed to give *greater accuracy* and they posses the advantage of requiring only the function values at some selected points on the subinterval.

Second order Runge-Kutta formula

If we set

$$k_1 = h\, f_0$$
$$k_2 = h\, f(x_0 + h,\ y_0 + k_1)$$

then

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$$

which is the *second order Runge-Kutta formula*. The error in this formula can be shown to be order h^3 by expanding both sides by Taylor's series.

Fourth order Runge-Kutta formula

Higher order Runge-Kutta formula exist, of which we mention only the *fourth order formulae*.



If
$$\left.\begin{aligned} k_1 &= h\, f(x_0, y_0) \\ k_2 &= h\, f\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) \\ k_3 &= h\, f\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) \\ k_4 &= h\, f\left(x_0 + h,\ y_0 + k_3\right) \end{aligned}\right\}$$

then $$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

in which the error is of order h^5.

Example

Given $\frac{dy}{dx} = y - x$, *where y(0) = 2. Find y(0.1) and y(0.2) correct to four decimal places.*

Solution

Runge-Kutta Fourth Order Formula

To determine y(0.1), we have $x_0 = 0$, $y_0 = 2$ and h = 0.1. We then obtain

$k_1 = 0.2$ $k_2 = 0.205$

$k_3 = 0.20525$ and $k_4 = 0.21053$

Hence $$y(0.1) = 2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2.2052$$

Proceeding similarly, we obtain y(0.2) = 2.4214.

Example (JNTUH 2018, 10 marks)

Given $\frac{dy}{dx} = 1 + y^2$, *where y = 0 when x = 0, find y(0.2), y(0.4) and y(0.6). Use fourth order Runge-Kutta method.*

Solution

We take h = 0.2. With $x_0 = y_0 = 0$, we obtain from (ix)

$k_1 = 0.2$

$k_2 = 0.2\ (1.01) = 0.202$

$k_3 = 0.2\ (1 + 0.010201) = 0.20204$

$k_4 = 0.2\ (1 + 0.040820) = 0.20816$



and $y(0.2) = 0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2027$, which is correct to four decimal places.

To compute y(0.4), we take $x_0 = 0.2$, $y_0 = 0.2027$ and h = 0.2. With these values (9) gives

$k_1 = 0.2\ [1 + (0.2027)^2\ ] = 0.2082$

$k_2 = 0.2\ [1 + (0.3068)^2\ ] = 0.2188$

$k_3 = 0.2\ [1 + (0.3121)^2\ ] = 0.2195$

$k_4 = 0.2\ [1 + (0.4222)^2\ ] = 0.2356$

and y(0.4) = 0.2027 + 0.2201 = 0.4228, which is correct to four decimal places.

Finally, taking $x_0 = 0.4$, $y_0 = 0.4228$ and h = 0.2, and proceeding as above,

we obtain y(0.6) = 0.6841.

Example (UTU 2014)

Use the classical Runge-Kutta formula of fourth order to find the numerical solution at x = 0.8 for

$$\frac{dy}{dx} = \sqrt{x + y,}\quad y(0.4) = 0.41$$

Assume the step lengthy h=0.2

Solution

For n = 0 and h = 0.2, we have

$$x_0 = 0.4, y_0 = 0.41$$

$$K_1 = h\ f(x_0, y_0) = 0.2(0.4 + 0.41)^{1/2} = 0.18$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}K_1\right) = 0.2\left[0.4 + 0.1 + 0.41 + \frac{1}{2}(0.18)\right]^{1/2} = 0.2$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}K_2\right) = 0.2\left[0.4 + 0.1 + 0.4 + \frac{1}{2}(0.2)\right]^{1/2} = 0.2009975$$

$$K_4 = hf\left(x_0 + h, y_0 + K_3\right) = 0.2\left[0.4 + 0.2 + 0.41 + 0.2009975\right]^{1/2} = 0.2200906$$

$$y_1 = y_0 + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right) = 0.41 + o.2003476 = 0.6103476 = y(0.6)$$

For n =1, $x_1 = 0.4 + h = 0.4 + 0.2 = 0.6$, we obtain

$K_1 = 0.2200315$ [$K_1 = h\ f(x_1, y_1)$; x_0 , y_0 are replaced by x_1 , y_1]

$K_2 = 0.2383579$

$K_3 = 0.2391256$



$K_4 = 0.2568636$

$y_2 = 0.6103476 + 0.2386436 = o.8689912 = y\,(0.8)$

Hence, we have

$$y(0.6) \approx 0.61035, \quad y(0.8) \approx 0.84899$$

Problem

Use Runge-Kutta method to solve $\frac{dy}{dx} = xy$ *for x = 1.2, initially x = 1 and y = 2. Take h = 0.2.*

Answer: 2.4921

Problem

Apply the fourth order Runge-Kutta method to solve $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$. *Take step size h = 0.1 and determine y(0.1) and y(0.2) correct to four decimal places.*

Answer: y (0.1) = 1.1115, y (0.2) = 1.2530

Problem

Use the Runge-Kutta fourth order method to find the value of y when x=1 given that y=1 when x=0 and that $\frac{dy}{dx} = \frac{y-x}{y+x}$

Answer: 1.4983

MILNE'S SERIES METHOD

The predicted value of the solution is given by

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y')$$

The predicted value is substituted in corrector formula

$$y^{(1)}_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$

Example

Solve initial value problem $\frac{dy}{dx} = 1 + xy^2$, *y(0) = 1, h = 0.1 for x = 0.4 using Milne's method when it is given*



x	*0.1*	*0.2*	*0.3*
y	*1.105*	*1.223*	*1.355*

Solution

Here $f(x,y) = 1 = xy^2$

$x_0 = 0$	$y_0 = 1$	$y_0' = 1 + 0 \times 1^2 = 1$
$x_1 = 0.1$	$y_1 = 1.105$	$y_1' = 1 + (0.1)(1.105)^2 = 1.1221$
$x_2 = 0.2$	$y_2 = 1.223$	$y_2' = 1 + (0.2)(1.223)^2 = 1.2991$
$x_3 = 0.3$	$y_3 = 1.355$	$y_3' = 1 + (0.3)(1.355)^2 = 1.5508$

By the predictor formula, we have

$$y_4 = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$= 1 + \frac{4x(0.1)}{3}[2x1.221 - 1.2991 + 2x1.5508] = 1.539$$

Now $$y_4' = 1 + x_4 y_4^2 = 1 + (0.4)(1.539)^2 = 1.9474$$

Now by corrector formula

$$y_4 = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 1.223 + \frac{0.1}{3}[1.2991 + 4x1.5508 + 1.9474] = 1.53799$$

Corrected $$y_4' = 1 + x_4 y_4^2 = 1 + (0.4)(1.538)^2 = 1.9461$$

SYSTEM OF LINEAR EQUATIONS

Note: It is assumed that the student has basic knowledge of Matrices.

Gaussian Elimination Method

This is the elementary elimination method and it reduces the system which can be solved by *back substitution*. A although quite general, we shall describe this method by considering a system of three equations for the sake of clarity and simplicity.

Let the system be

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \qquad (1)$$



$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

We first form the augmented matrix of the system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \quad (2)$$

To eliminate x_1 from the second equation, we multiply the first equation by $-\frac{a_{21}}{a_{11}}$ and then add it to the second equation. Similarly, to eliminate x_1 from the third equation, we multiply the first equation by $-\frac{a_{31}}{a_{11}}$ and it to the third. This procedure can be shown thus

$$\begin{matrix} -\frac{a_{21}}{a_{11}} \\ -\frac{a_{31}}{a_{11}} \end{matrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \quad (3)$$

Where $-\frac{a_{21}}{a_{11}}$ and $-\frac{a_{31}}{a_{11}}$ are called the multipliers for the first stage of elimination. In this stage, we assumed that $a_{11} \neq 0$. The first equation is called the pivotal equation and a_{11} is called the first stage, the augmented matrix (3) becomes:

$$-\frac{a'_{32}}{a'_{22}} \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{bmatrix} \quad (4)$$

Where a'_{22}, a'_{23},…… are all changed elements. a'_{22} is the new pivot and the multiplier is $\frac{-a'_{32}}{a'_{22}}$. At the end of the second stage, we have the upper triangular system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{bmatrix} \quad (5)$$

from which the values of x_1, x_2 and x_3 can be obtained by back substitution.



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It is clear that the method will fail if one of the elements a_{11}, a'_{22} or a''_{33} vanishes. In such a case, the method can be modified by rearranging the row so that the pivot is nonzero. This pivoting and can be easily implemented on a computer. If this is impossible, then the matrix is singular and the equations have no solution. In particular, the number of nonzero diagonal elements in (5) will represent the rank of the original matrix.

GAUSS JORDAN METHOD

Instead of eliminating x_2 only in the third equation, we could have eliminated it from the first equation also so that at the end of the second stage, the augmented matrix becomes

$$\left[\begin{array}{cccc} a'_{11} & 0 & a'_{12} & b'_1 \\ \hline 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a'_{33} & b''_3 \end{array}\right] \qquad (6)$$

from which the values of x_1, x_2, and x_3 can be obtained directly without further computation. This modification of the Gaussian elimination is called the Gauss-Jordan method.

Example

Solve the following system by Gauss elimination and Gauss Jorden method

$$2x + y + z = 10$$

$$3x + 2y + 3x = 18$$

$$x + 4y + 9z = 16$$

Solution

Gauss Method

In this first stage, the multipliers are – 3/2 and – ½. We multiply the first equation successively by –3/2 and –1/2 add it to the second and third equations respectively to obtain the equations respectively to obtain the equations

$$\frac{1}{2}y + \frac{3}{2}z = 3$$

and

$$\frac{7}{2}y + \frac{17}{2}z = 11$$

The augmented matrix therefore becomes

$$\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/2 & 11 \end{bmatrix}$$



At the second stage, we eliminate y from the third equation by multiplying the second equation by –7 and adding it to the third. The resulting system will be upper triangular

$$2x + y + z = 10$$

$$\frac{1}{2}y + \frac{3}{2}z = 3$$

$$-2z = -10$$

Back substituting gives the solution:

$$x = 7\ ;\ y = -9\ ; z = 5$$

Gauss-Jordan Method

At the end of the first stage, we have as in (a) above

$$\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/2 & 11 \end{bmatrix}$$

Now, instead of eliminating y from the third equation only, we shall eliminate it from the first equation also. This reduces the system to

$$-7x + 0y + 5z = -24$$

$$+\frac{1}{2}y + \frac{3}{2}z = 3$$

$$-2z = -10$$

The elimination is now trivial and gives the same result as before.

Problem

Solve the system

$$5x - 2y + z = 4$$
$$7x + y - 5z = 8$$
$$3x + 7y + 4z = 10$$

by Gaussian Elimination Method and Gauss Jorden method.

Answer: $x = \frac{122}{9};\quad y = \frac{284}{327};\quad z = \frac{46}{327}$

Modification of the Gauss Method to compute the inverse

We know that X will be the inverse of A if



$$AX = I \qquad (1)$$

Where I is the unit matrix of the same order as A. It is required to determine the elements of X such that eq (1) is satisfied. For example, for third order matrices eq (1) may be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The reader can easily see that this equation is equivalent to the three equations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can therefore apply the Gaussian elimination method to each of these systems and the result in each case will be the corresponding column of A^{-1}. since the matrix of coefficients is the same in each case, we can solve all the three systems simultaneously. Starting with the augmented system

$$\left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

we obtain at the end of the first and second stages respectively

$$\left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ 0 & a'_{22} & a'_{23} & -a_{21}/a_{11} & 1 & 0 \\ 0 & a'_{32} & a'_{33} & -a_{31}/a_{11} & 0 & 1 \end{array}\right]$$



and $$\left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ 0 & a'_{22} & a'_{23} & \alpha_{21} & 1 & 0 \\ 0 & 0 & a''_{33} & \alpha_{31} & \alpha_{32} & 1 \end{array}\right]$$

Where $\alpha_{21} = -\frac{a_{21}}{a_{11}}, \quad \alpha_{31} = \frac{a_{21}}{a_{11}}\frac{a'_{32}}{a'_{22}} - \frac{a_{31}}{a_{11}}, \quad \alpha_{32} = -\frac{a'_{32}}{a'_{22}}$

The inverse can now be obtained easily, since the back substitution process with each column of the matrix **I** will yield the corresponding column of A^{-1} where **I** is given by

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha_{21} & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 1 \end{bmatrix}$$

Example

We shall consider again the system given in previous. We have here

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

The augmented system is

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$$

After the first stage, this becomes

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1/2 & 3/2 & -3/2 & 1 & 0 \\ 0 & 7/2 & 17/2 & -1/2 & 0 & 1 \end{array}\right]$$

Finally, at the end of the second stage, the system becomes:

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1/2 & 3/2 & -3/2 & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right]$$

This is equivalent to the three systems



$$\left[\begin{array}{ccc:c} 2 & 1 & 1 & 1 \\ 0 & 1/2 & 3/2 & -3/2 \\ 0 & 0 & -2 & 10 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 2 & 1 & 1 & 0 \\ 0 & 1/2 & 3/2 & 1 \\ 0 & 0 & -2 & -7 \end{array}\right]$$

and $$\left[\begin{array}{ccc:c} 2 & 1 & 1 & 0 \\ 0 & 1/2 & 3/2 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

Whose solution by back substitution yields the three column of the matrix

$$\begin{bmatrix} -3 & 5/2 & -1/2 \\ 12 & -17/2 & 3/2 \\ -5 & 7/2 & -1/2 \end{bmatrix}$$

Which is the required inverse A^{-1}.

Problem

Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 2 & 2 \end{bmatrix}$ *using Gaussian Eliminations.*

Answer: $A^{-1} = \begin{bmatrix} 1.2 & -0.4 & 0.2 \\ -0.2 & -0.1 & 0.3 \\ -0.4 & 0.3 & 0.1 \end{bmatrix}$

LU DECOMPOSITION METHOD

This method is based on the fact that square matrix A can be factorized into the form LU where L is unit lower triangular and U is upper triangular, if all the principal minors of A are non-singular, i.e., if

$$a_{11} \neq 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \quad \text{etc..}$$

It is a standard result of linear algebra that such a factorization, when it exists, is unique.



We consider, for definiteness, the linear system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Which can be written in the form

$$AX = B \quad (1)$$

Let $A = LU$ (2)

Where $$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad (3)$$

And $$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad (4)$$

Hence eq (1) becomes

$$LUX = B \quad (5)$$

If we set

$$UX = Y \quad (6)$$

Then eq. (5) may be written as

$$LY = B \quad (7)$$

Which is equivalent to the system

$$y_1 = b_1$$
$$l_{21}y_1 + y_2 = b_2$$
$$l_{31}y_1 + l_{32}y_2 + y_3 = b_3$$

and can therefore be solved for y_1, y_2, y_3 by the *forward substitution*. When Y is known, the system eq (6) becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$
$$u_{22}x_2 + u_{23}x_3 = y_2$$
$$u_{33}x_3 = y_3$$

Which can be solved for x_1, x_2, x_3 by the *backward substitution*.



We shall now describe a scheme for computing the matrices L and U, and illustrate the procedure with a matrix of order #. Form the relation eq (2), we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and equating the corresponding elements of both sides, we get

$$\left.\begin{array}{ll} u_{11} = a_{11} & u_{12} = a_{12},\ u_{13} = a_{13} \\ l_{21}u_{11} = a_{21} & l_{31}u_{11} = a_{31} \\ \text{or} \quad l_{21} = \dfrac{a_{21}}{a_{11}}; & \text{or} \quad l_{31} = \dfrac{a_{31}}{a_{11}} \\ l_{21}u_{12} + u_{22} = a_{22} & l_{21}u_{13} + u_{23} = a_{23} \\ \text{or} \quad u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}}a_{12}; & \text{or} \quad u_{23} = a_{23} - \dfrac{a_{21}}{a_{11}}a_{13} \\ l_{31}u_{12} + l_{32}u_{22} = a_{32} & \\ \text{or} \quad l_{32} = \dfrac{a_{32} - \dfrac{a_{31}}{a_{11}}a_{12}}{u_{22}} & \\ \text{and lastly,} \quad l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33} & \end{array}\right\} \quad (8)$$

from which u_{33} can be computed.

We thus have a systematic procedure to evaluate the elements of L and U. First, we determine the first row of U and the first column of L; then we determine the second row of U and the second column of L, and finally, we compute the third row of U. The procedure can be obviously generalized.

When the factorization is effected, the inverse of A can be computed from the formula

$$A^{-1} = (LU)^{-1} = U^{-1}\, L^{-1} \quad (9)$$

Example

Solve the equation

$$2x + 3y + z = 9$$

$$x + 2y + 3z = 6$$

$$3x + y + 2z = 8$$

by the factorization method.



Solution

We have $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

Clearly, $u_{11} = 2, \quad u_{12} = 3, \quad u_{13} = 1$

Also $l_{21}u_{11} = 1$ so that $l_{21} = \frac{1}{2}$

And $l_{31}u_{11} = 3$ so that $l_{31} = \frac{3}{2}$

For u_{22} and u_{23}, w have the equations

$$l_{21}u_{12} + u_{22} = 2 \quad \text{and} \quad l_{21}u_{13} + u_{23} = 3$$

from which we obtain

$$u_{22} = \frac{1}{2} \quad \text{and} \quad u_{23} = \frac{5}{2}$$

Finally, l_{32} and u_{33} are obtained from

$$l_{31}u_{12} + l_{32}u_{22} = 1 \quad \text{and} \quad l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$$

and hence $l_{32} = -7$ and $u_{33} = 18$

It follows that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$

from eq (2)

And hence the given system of equations can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

from eq (5)



or as $$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

from eq (7)

Solving this system by forward substitution, we get

$$y_1 = 9$$

$$\frac{1}{2} y_1 + y_2 = 6 \quad \text{or} \quad y_2 = \frac{3}{2}$$

and $$\frac{3}{2} y_1 - 7y_2 + y_3 = 8 \quad \text{or} \quad y_3 = 5$$

Hence the solution of the original system is given by

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

from eq (6)

which, when solved by the back substitution process, gives

$$x = \frac{35}{18}; \quad y = \frac{29}{18}; \quad z = \frac{5}{18}$$

Problem

Factorized the matrix $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ *into the form LU, where is unit lower triangular and U is upper triangular, and hence solve the system of equation.*

Answer: $L = \begin{bmatrix} 1 & 0 & 0 \\ 7/5 & 1 & 0 \\ 3/5 & 41/19 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 19/5 & -32/5 \\ 0 & 0 & 327/19 \end{bmatrix}$

ITERATIVE METHOD (JACOBI & GAUSS SEIDEL METHODS)

Let the system be given by



$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2 \\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ... + a_{3n}x_n = b_3 \\
&..... .... \\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ... + a_{nm}x_n = b_n
\end{aligned} \qquad (1)
$$

in which the diagonal elements a_{ii} do not vanish. If this is not the case, then the equations should be rearranged so that this condition is satisfied. Now, we rewrite the system (1) as

$$
\begin{aligned}
x_1 &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - .... - \frac{a_{1n}}{a_{11}}x_n \\
x_2 &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{22}}x_2 - .... - \frac{a_{2n}}{a_{22}}x_n \\
x_3 &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 - .... - \frac{a_{3n}}{a_{33}}x_n \\
&..... .... \\
x_n &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_2 - .... - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}
\end{aligned} \qquad (2)
$$

Suppose $x_1^{(1)}$, $x_2^{(1)}$……$x_n^{(1)}$ are any first approximations to the unknowns x_1, x_2,….. x_n. Substituting in the right side of (2), we find a system of second approximation.

$$
\begin{aligned}
x_1^{(2)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(1)} - \frac{a_{13}}{a_{11}}x_3^{(1)} - .... - \frac{a_{1n}}{a_{11}}x_n^{(1)} \\
x_2^{(2)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(1)} - \frac{a_{23}}{a_{22}}x_2^{(1)} - .... - \frac{a_{2n}}{a_{22}}x_n^{(1)} \\
x_3^{(2)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x_1^{(1)} - \frac{a_{32}}{a_{33}}x_2^{(1)} - .... - \frac{a_{3n}}{a_{33}}x_n^{(1)} \\
&..... .... \\
x_n^{(2)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(1)} - \frac{a_{n2}}{a_{nn}}x_2^{(1)} - .... - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}^{(1)}
\end{aligned} \qquad (3)
$$

Similarly, if $x_1^{(n)}$, $x_2^{(n)}$…..$x_n^{(n)}$ are a system of nth approximations, then the next approximation is given by the formula



$$x_1^{(n+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(n)} - \frac{a_{13}}{a_{11}} x_3^{(n)} - .... - \frac{a_{1n}}{a_{11}} x_n^{(n)}$$

$$x_2^{(n+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(n)} - \frac{a_{23}}{a_{22}} x_2^{(n)} - .... - \frac{a_{2n}}{a_{22}} x_n^{(n)}$$

$$x_3^{(n+1)} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(n)} - \frac{a_{32}}{a_{33}} x_2^{(n)} - .... - \frac{a_{3n}}{a_{33}} x_n^{(n)} \qquad (4)$$

....

$$x_n^{(n+1)} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n)} - \frac{a_{n2}}{a_{nn}} x_2^{(n)} - .... - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(n)}$$

If we write eq (2) in the matrix form

$$X = BX + C \qquad (5)$$

Then the iteration formula eq..(20) may be written as

$$X^{(n+1)} = BX^{(n)} + C \qquad (6)$$

This method is due to **Jacobi.**

It can be shown that a *sufficient* condition for the convergence of this method is that

$$\|B\| < 1 \qquad (7)$$

A simple modification of this method sometimes yields faster convergence and is described below.

Suppose $x_1^{(1)}$, $x_2^{(1)}$……$x_n^{(1)}$ are any first approximations to the unknowns x_1, x_2,….. x_n.

In the first equation of eq (2), we substitute the first approximation ($x_1^{(1)}, x_2^{(1)}, x_3^{(1)}$ .....$x_n^{(1)}$ into the right hand side and denote the result as $x_1^{(2)}$. In the second equation we substitute $(x_1^{(2)}, x_2^{(1)}, x_3^{(1)}$ .....$x_n^{(1)})$ and denote the result as $x_2^{(2)}$ .In the third we substitute $(x_1^{(2)}, x_2^{(2)}, x_3^{(1)}$ .....$x_n^{(1)})$ and call the result as $x_3^{(2)}$.

In this manner, we complete the first stage of iteration and the entire process is repeated till the values of $x_1, x_2, \ldots\ldots x_n$ are obtained to the accuracy required. It is clear, therefore, that this method uses an improved component as soon as it is available and it is called ***Seidel method***.

Example

Solve



$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

Solution

To solve these equations by iterative method, we rewrite them as follows :

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3$$

Gauss-Seidel Method

n	x_1	X_2	x_3	x_4
1	0.3	1.56	2.886	-0.1368
2	0.8869	1.9523	2.9566	-0.0248
3	0.9836	1.9899	2.9924	-0.0042
4	0.9968	1.9982	2.9987	-0.0008
5	0.9994	1.9997	2.9998	-0.0001
6	0.9999	1.9999	3.0	0.0
7	1.0	2.0	3.0	0.0

Jacobi's Method

n	x_1	x_2	x_3	x_4	Remarks
1	0.3 $x_1^{(1)}$	1.5 $x_2^{(1)}$	2.7 $x_3^{(1)}$	-0.9 $x_4^{(1)}$	Proceed with assumed values of $x_1^{(1)}$,$x_2^{(1)}$,$x_3^{(1)}$, and $x_4^{(1)}$ as 0.3,1.5,2.7 and –0.9. We get values given in next row.



2	0.78 $x_1^{(2)}$	1.74 $x_2^{(2)}$	2.7 $x_3^{(2)}$	-0.18 $x_4^{(2)}$	Proceed with values given in this row. We get $x_1^{(3)}$, $x_2^{(3)}$ etc. given in next row.
3	0.9 $x_1^{(3)}$	1.908 $x_2^{(3)}$	2.916 $x_3^{(3)}$	-0.108 $x_4^{(3)}$	Proceed with values given in this row. We get $x_1^{(4)}$, $x_2^{(4)}$ etc. given in next row.
4	0.9624	1.9608	2.9592	-0.036	
5	0.9845	1.9848	2.9851	-0.0158	
6	0.9939	1.9938	2.9938	-0.006	
7	0.9975	1.9975	2.9976	-0.0025	
8	0.9990	1.9990	2.9990	-0.0010	
9	0.9996	1.9996	2.9996	-0.0004	
10	0.9998	1.9998	2.9998	-0.0002	
11	0.9999	1.9999	2.9999	-0.0001	
12	1.0	2.0	3.0	0.0	

Problem

Solving the following systems, viz.,

(a)
$$\begin{aligned} 10x + 2y + z &= 9 \\ 2x + 20y - 2z &= -44 \\ -2x + 3y + 10z &= 22 \end{aligned}$$

(b)
$$\begin{bmatrix} 17 & 65 & -13 & 50 \\ 12 & 16 & 37 & 18 \\ 56 & 23 & 11 & -19 \\ 3 & -5 & 47 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 84 \\ 25 \\ 36 \\ 18 \end{bmatrix}$$

by (i) Jacobi's method, and (ii) Gauss-Seidel method. In each case, carry your computations to two decimal places, and proceed upto 10 iterations.

Answer: (a) $x = 1.0,\ y = -2.0,\ z = 3.0$

(b) $x_1 = 4.84,\ x_2 = -4.70,\ x_3 = -1.64,\ x_4 = 5.72$

EIGENVALUE AND EIGENVECTOR

Let A be a square matrix of order n with elements a_{ij}. We wish to find a column vector X and a constant λ such that

$$A X = \lambda X \qquad (1)$$



In equation (1), λ is called the ***eigenvalue*** an X is called the corresponding ***eigenvector***.

The matrix eq..(1), when written out in full, represents a set of homogeneous linear equations

$$\left.\begin{aligned}(a_{11}-\lambda)x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n &= 0\\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \ldots\ldots\ldots + a_{2n}x_n &= 0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &.\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &.\\ a_{n1}x_1 + a_{n2}x_z + \ldots\ldots\ldots + (a_{nn}-\lambda)x_n &= 0\end{aligned}\right\} \quad (2)$$

A nontrivial solution exists only when the coefficient determinant in eq..(2) vanishes. Hence, we have

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0 \quad (3)$$

This equation, called the characteristic equation of the matrix A, is a polynomial equation of degree n in λ, the polynomial being called the characteristic polynomial of A. If the roots of eq..(3) be given by λ_i (i = 1,2,…….n), then for each value of λ_i, there exists a corresponding X_i such that

$$A\ X_i = \lambda_i\ X_i \quad (4)$$

The eigen values are obtained by solving the algebraic equation (3).

Example

Find the characteristic equation for the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{pmatrix}$$

hence find the eigenvalues and corresponding eigenvectors of A.

Solution

Here $A - \lambda I = \begin{pmatrix} 2-\lambda & 1 & -1 \\ 0 & -2-\lambda & -2 \\ 1 & 1 & -\lambda \end{pmatrix}$

so $\det\| a - \lambda I \| = -\lambda^3 + \lambda$ [after simplification]



Thus the characteristic equation, $\det(a-\lambda I) = 0$, is just $\lambda^3-\lambda = 0$ and hence the eigenvalues, the roots of this equation are given by

$$\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1$$

The eigenvectors corresponding to these eigenvalues may now be calculated in turn.

(1) $\lambda_1 = 0$

We have to solve $(A-\lambda I)x = 0$ for x to obtain $x^{(1)}$. But

$$A- \lambda_1 I = \begin{pmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{pmatrix}$$

and so the system of equations to be solved is just

$$2x_1 + x_2 - x_3 = 0$$

$$-2x_2 - 2x_3 = 0$$

$$x_1 + x_2 = 0$$

These yield immediately $x_1 = -x_2 = x_3$ and hence the solution of the system is given by $x_1 = k$, $x_2 = -k$, $x_3 = k$ for any k. Thus an eigenvector is given by

$$\begin{pmatrix} k \\ -k \\ k \end{pmatrix}$$

and it will be normalized if we choose k such that $k^2 + k^2 + k^2 = 1$, i.e. $k = 1/\sqrt{3}$.

Thus the normalized eigenvector corresponding to λ_1 is given by

$$x^{(1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(2) $\lambda_2 = 1$

Solve $(A - \lambda_2 I)x = 0$

We get $x_1 = (5/3)x_3$, $x_2 = (-2/3)x_3$



$$\therefore \text{ eigenvector} = \begin{pmatrix} \frac{5}{3}\,k \\ -\frac{2}{3}\,k \\ k \end{pmatrix}$$

It will be normalized if we choose k such that

$$\frac{25}{9}\,k^2 + \frac{4}{9}\,k^2 + k^2 = 1 \quad \text{i.e. } k = 3/\sqrt{38}\,.$$

Thus the normalized eigenvector corresponding to λ_2 is given by

$$x^{(2)} = \frac{1}{\sqrt{38}} \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

(3) $\lambda_3 = -1$

do yourself.

$$x^{(3)} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Problem

Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Answer: -2, 4, 6; $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

Problem

Determine the eigenvalues and the corresponding eigenvectors for the given matrix

$$\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$$



Answer: $\lambda_1 = 0,\ \lambda_2 = 3, \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}, \quad \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$



ASSIGNMENT

NUMERICAL DIFFERENTIATION

Q.1. (AKTU 2019, 7 marks): From the following table of values of x and y, obtain dy/dx for x = 1.2, 1.6, 2.2.

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Answer: 3.3205; 4.9530; 9.0228

Q.2. (AKTU 2018, 2020, 2021, 7 marks): Find x for which y is maximum and find the max value of y

x	1.2	1.3	1.4	1.5	1.6
y	0.9320	0.9636	0.9855	0.9975	0.9996

Answer: Solved in this module.

Q.3. (KU 2019, 7.5 marks): Find the minimum value of y(x) from the function tabulated below:

x	0.60	0.65	0.70	0.75
y(x)	0.6221	0.6155	0.6138	0.6170

Answer: Solved in this module.

Q.4. (GTU NM 2020, 4 marks): Find the first derivative at x = 1.1 from the following table

x	1	1.2	1.4	1.6	1.8	2.0
f(x)	0	0.1280	0.5440	1.2960	2.4320	4.0000

Answer: 0.630

Q.5. (UTU 2013): Find dy/dx at x = 0.1 from the following table

x	0.1	0.2	0.3	0.4
y	0.9975	0.9900	0.9776	0.9604

NUMERICAL INTEGRATION

Q.6. (JNTUH 2018, 2 marks): Write Simpson's 1/3rd and 3/8th rule formulas.

Q.7. (UTU 2014): State and prove Simpson's 3/8th rule with the help of quadrature formula and discuss its nature.

Q.8. (AKTU 2018, 2019, 7 marks): Apply Simpson's 3/8th rule to obtain approximate value of

$(i) \int_0^{\pi/2} e^{\sin x} dx$

$(ii) \int_0^{0.3} (2x - x^2)^{1/2} dx$ using Simpson's rule with 6 intervals.

Answer: Solved in this module.

Q.9. (AKTU 2019, UTU 2015, 3.5 marks): Using (i) Simpson's 1/3 rule (ii) Simpson's 3/8th rule on integration evaluate $\int_0^6 \frac{1}{1+x} dx$



Answer: Solved in this module.

Q.10. (UTU 2017, 10 marks): Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by trapezoidal rule, Simpson's 3/8 rule and Simpson's 1/3rd rule by dividing the interval into 9 parts. Compare the results with its actual value.

Answer: A similar problem is solved in this module.

Q.11. (AKTU 2021, 7 marks): State and prove Simpson one third rule and evaluate $\int_0^2 \frac{1}{x^2+x+1}dx$ using Simpson's rule with eight intervals.

Q.12. (BPTU 2020, 6 marks): Evaluate the integral $\int_2^3 \frac{\sin x}{1-x^2}dx$ by using Simpson's 1/3rd rule.

Q.13. (GTU NM 2020, UTU 2013, 7 marks): Use Simpson's 1/3rd and Simpson's 3/8 rule to evaluate the following integral

$$\int_0^1 \frac{1}{1+x^2}dx$$

Answer: 0.785397 and 0.785395

Q.14. (JNTUK NM 2020, 7 marks): Evaluate $\int_1^2 (x^2+1)dx$ by (i) Simpson's 1/3 rule (ii) Simpson's 3/8 rule.

Q.15. (KU 2019, GTU 2014, 7.5 marks): Using Simpson's 1/3rd rule evaluate $\int_0^{0.6} e^{-x^2}dx$ by taking seven ordinates.

Answer: 0.5351

Q.16. (BPTU 2020, JNTUH 2018, 2 marks): State trapezoidal rule to evaluate $\int_{x_0}^{x_n} f(x)dx$ and why is Trapezoidal rule so called?

Q.17. (GTU NM 2020, 4 marks): Use Trapezoidal rule to evaluate $\int_1^3 \frac{1}{x}dx$ taking subintervals.

Q.18. (JNTUK NM 2020, 2 marks): Evaluate $\int_0^1 \sin x dx$ using Trapezoidal rule (taking n = 5).

Q.19. (JNTUK NM 2021, 2 marks): Evaluate $\int_0^1 \frac{dx}{1+x}$ using Trapezoidal rule.

Answer: 0.6970 taking h = 0.25.

Q.20. (UTU 2016, 2 marks): Show that $\int_0^1 \frac{dx}{1+x}$ = log2 = 0.69315 using Simpson's 1/3rd rule.

Answer: 0.6932 taking h = 0.25

SOLUTION OF ODE



Q.21. (BPUT 2020, 2 marks): Which is better Taylor's method or Runge-Kutta method? why?

Answer: Runge-Kutta method is better since higher order derivatives of y are not required. Taylor series method involves use of higher order derivatives which may be difficult in case of complicated algebraic equations.

Q.22. (JNTUH 2018, 3 marks): What are the limitations of Taylor's series method?

Answer:

- Successive terms get very complex and hard to derive
- Truncation error tends to grow rapidly away from expansion point
- Almost always not as efficient as curve fitting or direct approximation

Q.23. (AKTU 2018, 2020, 2021, 7 marks): Given the initial value problem $\frac{dy}{dx} = x^3 - y^3, y(0) = 1$

Find the numerical solution of differential equation at x = 0.6 with h = 0.2 by using Runge-Kutta method of fourth order.

Q.24. (AKTU 2019, 7 marks): Using Runge-Kutta method of fourth order, find y(0.8) correct to 4 decimal places if $\frac{dy}{dx} = y - x^2$, y(0.6) = 1.7379 taking h = 0.1.

Q.25. (BPUT 2020, GTU NM 2020, 6 marks): Apply Runge-Kutta method to find approximate value of y for x = 0.2 in steps of 0.1 if $\frac{dy}{dx} = x + y^2$ given that y = 1 when x = 0.

Answer: y (0.2) = 1.2736

Q.26. (JNTUH 2018, 10 marks): Use Runge-Kutta method of order four to find y when x = 0.6 in steps of 0.2 given that $\frac{dy}{dx} = 1 + y^2, y(0) = 0$

Answer: Solved in this module.

Q.27. (JNTUH 2019, 3 marks): Find y(0.1) and y(0.2) using 4th order Runge-Kutta method given that y' = xy + y^2, y(0) = 1.

Q.28. (KU 2019, 7.5 marks): Find an approximate value of y(0.1) in the steps of 0.1, if $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, y = 0, y(0) = 1, y'(0) = 0 by Runge-Kutta method of fourth order.

Q.29. (GTU NM 2020, 7 marks): Solve using Runge-Kutta 4th order method.

$\frac{dy}{dx} = x + y$, y(0) = 1 using h = 0.05 for y(0.1).

Q.30. (JNTUK NM 2021, 7 marks): Find the solution of $\frac{dy}{dx} = x^2 + y$, y(0) = 1 at x = 0.02 using Runge-Kutta method of fourth order.

Q.31. (PTU NM 2018, 5 marks): Using Runge-Kutta method of order 4, find y(0.2) for the equation y' = (y - x)/(y + x), y(0) = 1, take h = 0.2.



Q.32. (PTU NM 2019, 5 marks): Use the Runge-Kutta method of order 4 to approximate the solution of the following initial value problem

$y' = y - t^2 + 1, 0 \le 2, y(0) = 0.5$.

Q.33. (UTU 2014): Use Runge-Kutta formula of fourth order to find the numerical solution at x = 0.8 for the differential equation $y' = \sqrt{x+y}, y(0) = 0.41$, assume the step size h = 0.2.

Answer: Solved in this module.

Q.34. (UTU 2015): Apply the Runge-Kutta method of fourth order to find approximate value of y for x = 0.1 and 0.2 if $\frac{dy}{dx} = \frac{4x}{y} - xy$, given that $y_0 = 3$ where $x_0 = 0$.

Q.35. (UTU 2016): Use Runge-Kutta method, find the solution of the differential equation $\frac{dy}{dx} = 3x + \frac{y}{2}$ with y(0) = 1 at x = 0.1.

Answer: 1.1672

Q.36. (UTU 2017): Solve $\frac{dy}{dx} = yz + x, \frac{dz}{dx} = xz + y$. Given that y(0) = 1, z(0) = -1 for y(0.1), z(0.1) and y(0.2), z(0.2).

Q.37. (BPUT 2020, 16 marks): Use Euler's method and Modified Euler's method with one step to find the value of y at x = 0.1 to five significant figures, where $\frac{dy}{dx} = x^2 + y$ and y = 0.94 when x = 0.

Answer: Solved in this module.

Q.38. (JNTUH 2019, 3 marks): Given y' = xy with y(0) = 1. Find y(0.2) with = 0.1 by Euler's method.

Q.39. (KU 2019, 7.5 marks): Given that $\frac{dy}{dx} = x + y$ and y = 0, x = 0. Find an approximate value of y at x = 0.3 by Euler's method taking h = 0.1.

Q.40. (GTU NM 2020, 3 marks): Use Euler's method with h = 0.1 to find the solution of the equation

$\frac{dy}{dx} = x^2 + y^2$, given that y(0) = 0 in the range $0 \le x \le 5$.

Q.41. (JNTUH 2018, 10 marks): Find the successive approximate solution of the differential equation y' = y, y(0) = 1 by Picard's method and compare it with exact solution.

Q.42. (AKTU 2019, 3.5 marks): Use Picard's method to obtain y for x = 0.2. Given $\frac{dy}{dx} = x - y$ with initial condition y = 1 when x = 0 correct up to four decimal places.

Answer: Solved in this module.

Q.43. (JNTUH 2019, 10 marks): Solve the equation $y' = x + y^2$ subject to condition y(0) = 1 by Picard's method.

Answer: Solved in this module.



Q.44. (GTU NM 2020, 7 marks): State the Picard's formula and solve the equation for x = 0.2.

$$\frac{dy}{dx} = x^2 - y; y(0) = 1$$

Q.45. (JNTUK NM 2020, 7 marks): Solve $\frac{dy}{dx} = \frac{xy}{2}$ using Picard's method for x = 1.1 given y(1) = 1.

Q.46. (JNTUK NM 2021, 7 marks): Find the solution of $\frac{dy}{dx} = x + y$, y(0) = 1 at x = 0 using Picard's method.

Q.47. (GTU NM 2020, 3 marks): use Taylor's series method to find approximate value of y(1.2) if

$$\frac{dy}{dx} = x + y; y(1) = 0$$

Answer: y(1.2) = 0.2428 taking h = 0.1.

Q.48. (GTU NM 2020, 7 marks): State the Taylor's method and solve equation

$$\frac{dy}{dx} = y - \frac{2x}{y}, y(0) = 1$$ for x = 0.1.

Answer: y(0.1) = 1.0954

Q.49. (GTU NM 2020, 7 marks): Using Milne's Predictor Corrector method to obtain y(0.4) by solving $\frac{dy}{dx} = 2e^x - y$ given that y(0) = 2, y(0.1) = 2.01, y(0.2) = 2.04, y(0.3) = 2.09.

SYSTEM OF LINEAR EQUATIONS

Q.50. (BPUT 2020, 2 marks): State the condition of the convergences for solving linear system of equation by Gauss-Seidel method.

Q.51. (BPUT 2020, 6 marks): Solve the following system of equation by using LU decomposition method.

4x + y + 3z = 11

3x + 4y +2z = 11

2x + 3y +z = 7

Q.52. (BPUT 2020, 16 marks): Solve by using Gauss-Jacobi method and Gauss Seidel method correct to 5 decimal places and compare which method is better:

10x + y + z = 12

x + 10y + z = 12

x + y + 10z = 12

Q.53. (AKTU 2018, 2020, 2021, 7 marks): Decompose $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ in the form LU, where L is lower triangular matrix and U is upper triangular matrix and hence solve the system of equations:



$5x - 2y + z = 4$

$7x + y - 5z = 8$

$3x + 7y + 4z = 10.$

Q.54. (PTU NM 2018, 2 marks): Find Eigen values of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Q.55. (AKTU 2019, 3.5 marks): Determine the largest Eigen value and corresponding eigen vector of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ till three approximations.

Q.56. (JNTUH 2018, 10 marks): Show that the Gauss-Seidel methods diverge for solving the system of equations

$2x + 3y + z = -1$

$3x + 2y + 2z = 1$

$x + 2y + 2z = 6$

Q.57. (JNTUH 2019, 10 marks): Solve the following system of equations by Gauss-Seidel method

$8x_1 + x_2 - x_3 = 8$

$2x_1 + x_2 + 9x_3 = 12$

$x_1 - 7x_2 + 2x_3 = -4$

Q.58. (GTU NM 2020, 7 marks): Solve the following system by Gauss-Seidal iteration method:

$27x + 6y - z = 85$

$6x + 15y + 2z = 72$

$x + y + 54z = 110$

Q.59. (GTU NM 2020, 3 marks): Use Gauss elimination solve

$x + 2y + z = 8$

$2x + 3y + 4z = 20,$

$4x + 3y + 2z = 16$

Q.60. (PTU NM 2018, 5 marks): Solve the following equations by elimination method

$2x + y + z = 10$, $3x + 2y + 3z = 18$ and $x + 4y + 9z = 16$.

Q.61. (PTU NM 2019, 5 marks): Apply Gauss Jordan method to find the inverse of the matrix

$\begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}$

