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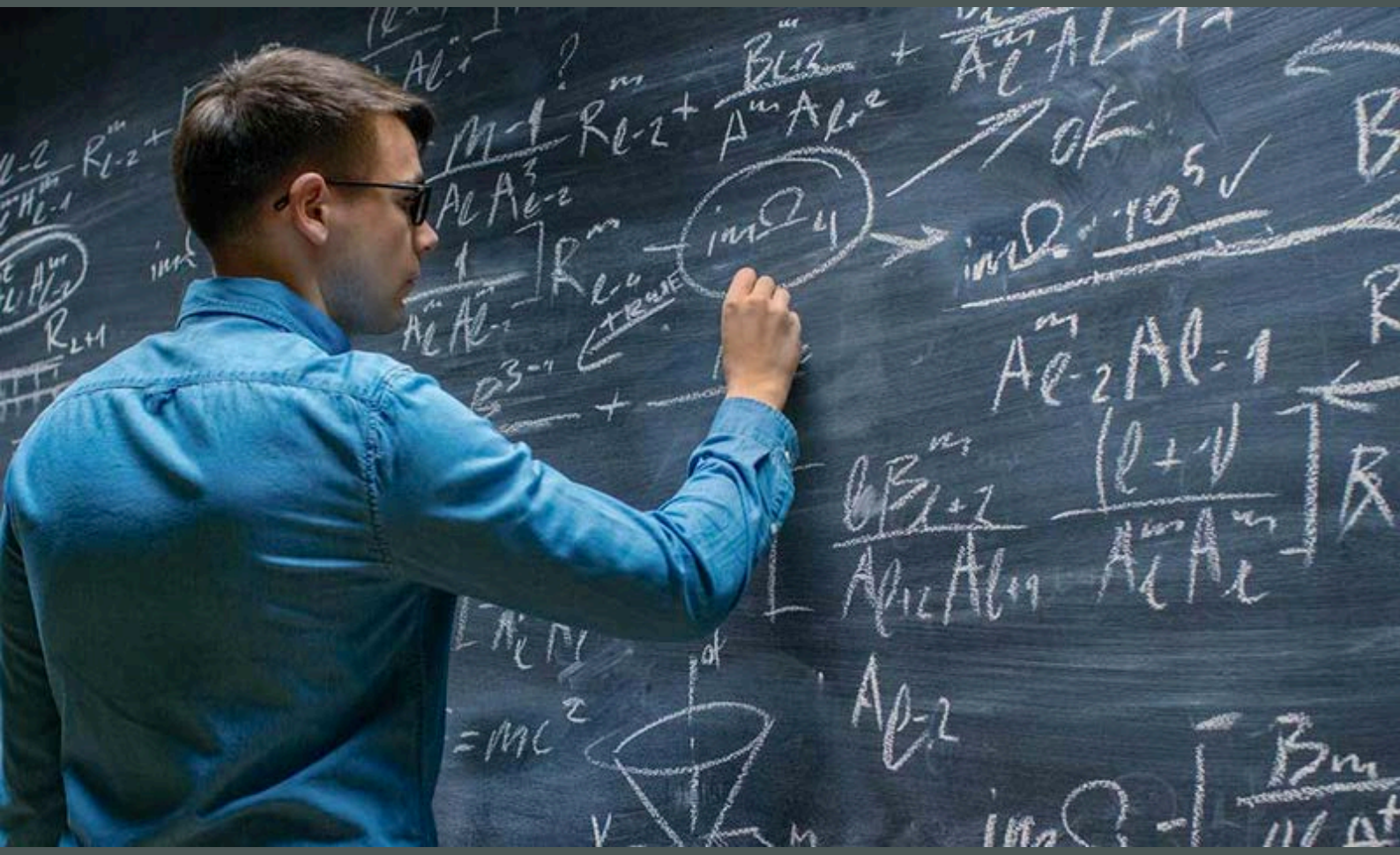

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# NUMERICAL METHODS - I

## NUMBERS AND THEIR ACCURACY

There are two kinds numbers, *exact* and *approximate* numbers. Examples of exact numbers are 1, 2, 3 ... 1/2, 3/2, .... √2, π, e ... etc. written in this manner. Approximate numbers are those that represent the numbers to a certain degree of accuracy. Thus, an approximate value of π is 3.1416, or if we desire a better approximation, it is 3.14159265. But we cannot write the exact value of π.

The digits that are used to express a number are called *significant digits* or *significant figures*. Thus the number 3.1416, 0.66667 and 4.0687 each contain five significant digits. The number 0.00023 has, however, only two significant digits, viz., 2 and 3, since the zeros serve only to fix the position of the decimal point.

Frequently, we come across number with a large number of digits, and it will be necessary to cut them to a usable number of figures. This process is called *rounding-off*. Generally, numbers are rounded-off according to the following rule

To round-off a number to n significant digits, discard all digits to the right of the nth digit, and if this discarded number is

(a) less than half a unit in the nth place, leave the nth digit unaltered
(b) greater than half a unit in the nth place, increase the nth digit by unity
(c) exactly half a unit in the nth place, increase the nth digit by unity if it is odd, otherwise leave it unchanged.

The number thus rounded-off is said to be correct to n significant figures.

### Example

The numbers given below are rounded-off to four significant figures:

1.6583 to 1.658

30.0567 to 30.06

0.859378 to 0.8594

3.14159 to 3.142





## ABSOLUTE, RELATIVE AND PERCENTAGE ERRORS

Absolute error is the numerical difference between the true value of a quantity and its approximate value. Thus, if $X$ is the true value of a quantity and $X_1$ s its approximate value, then the absolute error $E_A$ is given by

$$E_A = X - X_1 = \delta X \tag{1}$$

The relative error $E_R$ defined by

$$E_R = \frac{E_A}{X} = \frac{\delta X}{X} \tag{2}$$

and the percentage error $E_p$ by

$$E_p = 100 E_R \tag{3}$$

### Example (GTU NM 2020, 3 marks)

An approximate value of π is given by $X_1 = \frac{22}{7} = 3.1428571$ and its true value is $X = 3.1415926$. Find the absolute and relative errors. We have

$$E_A = X - X_1 = -0.0012645$$

and

$$E_R = \frac{-0.001264}{3.1415926} = -0.000402$$

## CONCEPT TO FIND ROOT OF EQUATION

In scientific and engineering work, a frequently occurring problem is to find the roots of equations of the form

$$f(x) = 0$$

If $f(x)$ is a quadratic, cubic or a biquadratic expression, then algebraic formulae are available for expressing the roots in terms of the coefficients. On the other hand when $f(x)$ is a polynomial of higher degree or an expression involving transcendental functions, e.g., $1 + \cos x - 5x, x \tan x - \cosh x, e^{-x} - \sin x, etc.,$ algebraic methods are not available, and recourse must be taken to find the roots by approximate methods.

In this chapter, we describe some numerical methods for the solution of equations of the form where $f(x)$ is algebraic or transcendental or a combination of both.





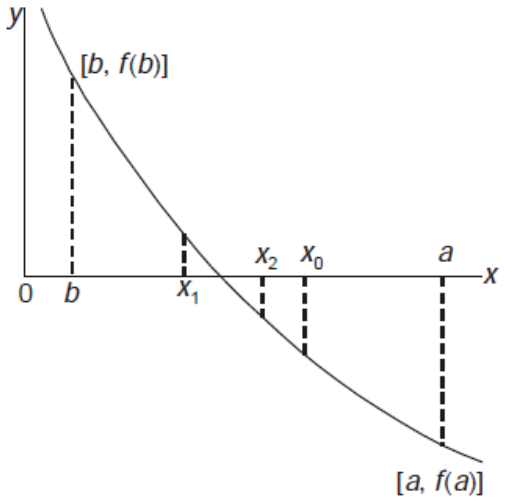
## THE BISECTION METHOD

If a function $f(x)$ is continuous between $a$ and $b$, $f(a)$ and $f(b)$ are of **opposite signs**, then there exists **at least one root** between $a$ and $b$.

For definiteness, let $f(a)$ be negative and $f(b)$ be positive. then the toot lies between $a$ and $b$ and let its **approximate value** be given $x_0 = \frac{(a+b)}{2}$.

If $f(x_{0)} = 0$, we conclude that $x_0$ is a **root** of the equation $f(x) = 0$. Otherwise, the root lies either between $x^0$ and $b$, or between $x^0$ and $a$ depending on whether $f(x_0)$ is negative or positive. We designate this new interval as $\lfloor a_1, b_1 \rfloor$ whose length is $|b-c|/2$. As before, this is **bisected** at $x_1$ and the new interval will be exactly half the length of the previous one. We **repeat** this process until the latest interval (which contains the root) is as small as desired, say $\varepsilon$.

The method is shown graphically in following figure




### Example

*Find a real root of the equation* $f(x) = x^3 - x - 1 = 0$

### Solution

Since $f(1)$ is negative and $f(2)$ positive a root lies between 1 and 2 and therefore we take $x_0 = \frac{2}{3}$.

Then $f(x_0) = \frac{27}{8} - \frac{3}{2} - \frac{15}{8}$, which is positive.

Hence the root lies between 1 and 1.5 and we obtain

$$x_1 = \frac{1+1.5}{2} = 1.25$$

We finds $f(x_1) = -\frac{19}{64}$,which is negative. We therefore conclude that the root lies between 1.25 and 1.5. It follows that

$$x_2 = \frac{1.25+1.5}{2} = 1.375$$

The procedure is repeated and the successive approximations are

$$x_3 = 1.3125, \quad x_4 = 1.34375, \quad x_5 = 1.328125, \text{ etc.}$$

## Example (GTU NM 2020, 4 marks)

*Find a real root of the equation* $x^3 - 2x - 5 = 0$

## Solution

Let $f(x) = x^3 - 2x - 5$

Then $f(2) = -1$ and $f(3) = 16$

Hence a root lies between 2 and 3 and we take

$$x_0 = \frac{2+3}{2} = 2.5$$

Since $f(x_0) = 5.6250$, we choose [2,2.5] as the new interval. Then $x_1 = \frac{2+2.5}{2} = 2.25$ and $f(x_1) = 1.890625$.

Proceeding in this way, the following table is obtained.

| $n$ | $a$ | $b$ | $x$ | $f(x)$ |
|---|---|---|---|---|
| 1 | 2 | 3 | 2.5 | 5.6250 |
| 2 | 2 | 2.5 | 2.25 | 1.8906 |
| 3 | 2 | 2.25 | 2.125 | 0.3457 |
| 4 | 2 | 2.125 | 2.0625 | -0.3513 |
| 5 | 2.0625 | 2.125 | 2.09375 | -0.0089 |
| 6 | 2.09375 | 2.125 | 2.10938 | 0.1668 |





| | | | | |
|---|---|---|---|---|
| 7 | 2.09375 | 2.10938 | 2.10156 | 0.07856 |
| 8 | 0.09375 | 2.10156 | 2.09766 | 0.03471 |
| 9 | 2.09375 | 2.09766 | 2.09570 | 0.01286 |
| 10 | 2.09375 | 2.09570 | 2.09473 | 0.00195 |
| 11 | 2.09375 | 2.09473 | 2.09424 | -0.0035 |
| 12 | 2.09424 | 2.09473 | | |

## Example (JNTUK NM 2021, 7 marks)

*Find a positive root of the equation* $xe^x - 1$, *which lies between 0 and 1.*

## Solution

Let $f(x) = ex^x - 1$

Since $f(0) = -1$ and $f(1) = 1.718$, it follows that a root lies between 0 and 1.

Thus, $x_0 = 0.5$.

Starting with this value and proceeding as in the previous example, the following table is constructed where only the sign of the function value is indicated.

| $n$ | $a$ | $b$ | $x$ | $f(x)$ |
|---|---|---|---|---|
| 1 | 0 | 1 | 0.5 | negative |
| 2 | 0.5 | 1 | 0.75 | positive |
| 3 | 0.5 | 0.75 | 0.625 | positive |
| 4 | 0.5 | 0.625 | 0.5625 | negative |
| 5 | 0.5625 | 0.625 | 0.59375 | positive |
| 6 | 0.5625 | 0.59375 | 0.5781 | positive |
| 7 | 0.5625 | 0.5781 | 0.5703 | positive |
| 8 | 0.5625 | 0.5703 | 0.5664 | negative |
| 9 | 0.5664 | 0.5703 | 0.5684 | positive |
| 10 | 0.5664 | 0.5684 | 0.5674 | positive |
| 11 | 0.5664 | 0.5674 | 0.5669 | negative |
| 12 | 0.5669 | 0.5674 | 0.5671 | negative |

The required root is 0.567 which is correct to 3 decimal places.

## Problem

*Obtain a root, correct to three decimal places, for each of the following equations using the bisection method.*

*1.* $x^3 + x^2 + x + 7 = 0$





*Answer: - 2.105*

*2.* $x^3 - 4x - 9 = 0$

*Answer.: 2.706*

## THE ITERATION METHOD

To describe this method for finding the roots of the equation $f(x) = 0$.

We rewrite this equation in the form

$$x = \phi(x) \qquad (1)$$

There are many ways of doing this.

For example, the equation

$$x^3 + x^2 - 1 = 0$$

can be expressed as

(a) $x = (1 + x)^{-1/2}$

(b) $x = (1 - x^3)^{1/2}$

(c) $x = (1 - x^2)^{1/3}$, etc.

Let $x_0$ be an approximate value of the desired root $\xi$. Substituting it for $x$ on the right side of (1), we obtain the first approximation

$$x_1 = \phi(x_0)$$

the successive approximations are then given by

$$x_2 = \phi(x_1),$$
$$x_3 = \phi(x_2),$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$x_n = \phi(x_{n-1})$$

When we express the equation $\phi(x) = 0$ in the form $x = \phi(x)$, then $\phi(x)$ must be such that $|\phi'(x)| < 1$ in an immediate neighbourhood of the root. It follows that if the initial approximation $x_0$ is chosen in an interval containing the root $\xi$ then the sequence of approximations converges to the root $\xi$.





## Order of convergence of iterative method

Convergence of an iterative method is judged by the order at which the error between successive approximations to the root decreases.

An iterative method is said to be kth order convergent if k is the largest positive real number, such that

$$\lim_{i\to\infty}\left|\frac{e_{i+1}}{e_i^k}\right| \le A$$

where A is a non-zero finite number called asymptotic error constant.

$e_i$ and $e_{i+1}$ are the errors in successive approximations, order convergence gives us the idea that in each iteration, the number of significant digits in each approximation increases k times.

### Example

*Find the root of the equation upto three decimal places* $2x = \cos x + 3$ *lying in the interval [3/2, π/2].*

### Solution

We rewrite the equation in the form

$$x = \frac{1}{2}(\cos x + 3)$$

so that $$\phi(x) = \frac{1}{2}(\cos x + 3)$$

We start with x = 3/2 = 1.5 radian

$$x_1 = \frac{1}{2}\left[\cos\left(1.5.\frac{180}{\pi}\right)^0 + 3\right] = 1.5354$$

$$x_1 = \frac{1}{2}\left[\cos\left(1.354.\frac{180}{\pi}\right)^0 + 3\right] = 1.5177$$

Similarly

$x_3 = 1.5265$

$x_4 = 1.5221$

$x_5 = 1.5243$

$x_6 = 1.5232$





$x_7 = 1.5238$

As $x_6$ and $x_7$ agree upto three decimal places, hence the root is 1.523.

### Example

*Use the method of iteration to find a positive root, between 0 and 1, of the equation $xe^x = 1$.*

### Solution

Writing the equation in the form

$$x = e^{-x}$$

We find that $\varphi(x) = e^{-x} = \frac{1}{e}$

Starting with $x_0 = 1$, we find that the successive iterates are given by

$x_1 = 1/e = 0.60653$

$x_2 = 0.54524$

$x_3 = 0.57970$

$x_4 = 0.56007$

and so on.

### Problem

*Use the method of iteration to find a root of the equation $\cos x = 3x - 1$*

*Answer: 0.6071*

### Problem

*Use the method of iteration to find a root of the equation $e^{-x} = 10x$*

*Answer.: 0.0913*

### Problem

*Use the method of iteration to find a root of the equation $e^x \tan x = 1$*

*Answer.: 0.5314*

## THE METHOD OF FALSE POSITION (REGULA-FALSI METHOD)

This is the **oldest method** for finding the real root of an equation, and closely resembles the bisection method. In this method, we choose two points $x_0$ and $x_1$ such that $f(x_0)$ and $f(x_1)$ are





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of opposite signs. Since the graph of $y = f(x)$ crosses the x-axis between these two points, a root must lie in between these points.

The second approximation to the root of $f(x) = 0$ is given by

$$x_2 = x_0 - \frac{f(x_0)}{f(x_1) - f(x_0)}(x_1 - x_0) \qquad (1)$$

If now $f(x_2)\, and\, f(x_0)$ are of opposite signs, then the root lies between $x_0\, and\, x_2$, and we replace $x_1\, by\, x_2$ in eq.(1), and obtain the next approximation. Otherwise, we replace $x_0\, and\, x_2$ and generate the next approximation.

The procedure is repeated till the root is obtained to the desired accuracy.

### Example (AKTU 2018, 2020, BPUT 2020, 7 marks)

*Find a real root of the equation*

$$f(x) = x^3 - 2x - 5$$

### Solution

We observe that

f(2) = -1 and f(3) = 16

and hence a root lies between 2 and 3. eq. (1) then gives

$$x_2 = 2 - \frac{(-1)}{16 - (-1)}(3 - 2) = 2.059$$

Hence the root lies between 2.059 and 3.0

Now f(2.059) = -0.386 and f(3) = 16

Using eq.(1) once again, we obtain

$$x_3 = 2.059 - \frac{(-0.386)}{16 - (-0.386)}(3 - 2.059) = 2.0812$$

Repeating the process, we obtain successively

$x_4 = 2.0904$

$x_5 = 2.0934$

and so on.





## Example

*Determine the real root of the equation $xe^x = 2$ using Regula-Falsi method correct to three decimal places.*

## Solution

$f(x) = xe^x - 2 = 0$

$f(0) = 0 - 2 = -2$

$f(0.5) = 0.5e^{0.5} - 2 = -1.1756$

$f(0.7) = -0.5903$

$f(0.9) = 0.2136$

Hence, one root lies between 0.7 and 0.9.

$f(x_0) = f(0.7) = -0.5903$

$f(x_1) = f(0.9) = 0.2136$

Now $$x_2 = x_0 - \frac{f(x_0)}{f(x_1) - f(x_0)}(x_1 - x_0)$$

$$= 0.7 - \frac{(-0.5903)}{0.2136 - (-0.5903)}(0.9 - 0.7)$$

$$= 0.8469$$

Now $f(x_2) = f(0.8469) = 0.8469e^{0.8469} - 2 = -0.0247$

Hence, root lies between 0.8469 and 0.9.

$\therefore$ $x_0 = 0.8469$; $x_1 = 0.9$

Which gives $f(x_0) = f(0.8469) = -0.0247$

and $f(x_1) = f(0.9) = 0.2136$

Using Regula-Falsi method

$$x_3 = 0.7 - \frac{(-0.0247)}{0.2136 - (-0.0247)}(0.9 - 0.8469)$$

$$= 0.8524$$

Similarly $x_4 = 0.8526$

Here $x_3$ is almost equal to $x_4$. We may stop here. Hence the desired root is 0.8526.

## Problem

*Find real root of the following equation by using Regula-Falsi method: $xe^x - 3 = 0$.*





*Answer: 1.046*

### Problem

*Given that the equation $x^{22} = 69$ has a root between 5 and 8. Use method of Regula-Falsi to determine it.*

*Answer: 6.8523651*

*Hint: $f(x) = x^{22} - 69$; $f(5) = -34.50675846$; $f(8) = 28.00586026$*

### Problem

Using Regula-Falsi method, obtain the real root of the following equation: $x^3 - 9x + 1 = 0$

Answer: 2.9428

## NEWTON-RAPHSON METHOD

This method is generally used to improve the result obtained by one of the previous methods. Let $x_0$ be an approximate root of $f(x) = 0$.

A better approximation than $x_0$ is therefore given by $x_1$, where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Successive approximations are given by $x_2, x_3, \ldots\ldots\ldots x_{n+1}$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (1)$$

Which is the *Newton-Raphson formula*.

## Order of convergence

Suppose xn differs from the root α by a small quantity $e_n$ so that

$$x_n = \alpha + e_n$$

and $$x_{n+1} = \alpha + e_{n+1}$$

Now $$e_{n+1} = e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_n)}$$

$$= e_n - \frac{f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \ldots.}{f'(\alpha) + e_n f''(\alpha) + \ldots}$$





$$= e_n - \frac{e_n f'(\alpha) + \frac{e_n^2}{2} f''(\alpha) + \ldots}{f'(\alpha) + e_n^2 f''(\alpha) + ..} \text{ [because f(α) = 0]}$$

$$= \frac{e_n^2 f''(\alpha)}{2[f'(\alpha) + e_n f''(\alpha)]} \quad \text{[neglecting higher powers of } e_n\text{)}$$

which finally converts to

$$e_{n+1} = \frac{e_n^2}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n^3}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 + \ldots.$$

or

$$\frac{e_{n+1}}{e_n^2} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 + \ldots.$$

$$\approx \frac{f''(\alpha)}{2f'(\alpha)} \quad \text{[neglecting terms containing powers of } e_n\text{]}$$

Hence by definition, the order of convergence of Newton-Raphson method is 2, i.e., Newton-Raphson method is **quadratic convergent**.

## Example (GTU NM 2020, 4 marks)

*Use the Newton-Raphson method to find a root of the equation: $x^3 - 2x - 5 = 0$.*

## Solution

Here $f(x) = x^3 - 2x - 5$

and $f'(x) = 3x^2 - 2$

Newton Raphson method gives

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

$f(1) = 1^3 - 2 \times 1 - 5 = 1 - 2 - 5 = -6$

$f(2) = 2^3 - 2 \times 2 - 5 = 8 - 4 - 5 = -1$

$f(3) = 3^3 - 2 \times 3 - 5 = +16$

So root lies between 2 and 3.

Choosing $x_0 = 2$, we obtain

$f(x_0) = 2^3 - 2(2) - 5 = -1$ and $f'(x_0) = 3(2)^2 - 2 = 10$.

Putting n = 0 in above formula





we obtain $x_1 = 2 - \left(\frac{-1}{10}\right) = 2.1$

Now $f(x_1) = 2.1^3 - 2(2.1) - 5 = 0.061$

and $f'(x_1) = 3(2.1)^2 - 2 = 11.23$

Hence $x_2 = 2.1 - \frac{0.061}{11.23} = 2.094568$

This example demonstrates that Newton-Raphson method converges *more rapidly* than the methods described in the previous sections.

## Example

*Find the real root of the equation, $x = e^{-x}$ using the Newton-Raphson method.*

## Solution

We write the equation in the form

$f(x) = xe^x - 1 = 0$

Then $f'(x) = e^x + xe^x = (1 + x)e^x$

$f(0) = -1$

$f(1) = 1 \times e^1 - 1 = e - 1$ (positive value)

Hence, root lies in between 0 and 1.

Let $x_0 = 1$

Then $x_1 = 1 - \frac{e-1}{2e} = \frac{1}{2}\left(1 + \frac{1}{e}\right)$

$= 0.6839397$

Now $f(x_1) = 0.3553424$

$f'(x_1) = 3.337012$

So that $x_2 = 0.6839397 - \frac{0.3553424}{3.337012} = 0.5774545$

Proceeding in this way, we obtain

$x_3 = 0.5672297$

and $x_4 = 0.5671433$

As $x_3$ and $x_4$ are almost same, we can stop here.





### Example

*Find the real root of tan x = 4x by using Newton-Raphson's method.*

### Solution

$f(x) = \tan x - 4x = 0$

$f'(x) = \sec^2 x - 4 = 0$

$f(0) = 0$ and $f(1) = -3.982$.

Now $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\tan x_n - 4x_n}{\sec^2 x_n - 4}$$

$$= \frac{x_n \sec^2 x_n - \tan x_n}{\sec^2 x_n - 4}$$

Take $x_0 = 0$ $$x_1 = \frac{x_0 \sec^2 x_0 - \tan x_0}{\sec^2 x_0 - 4} = \frac{0.1 - 0}{1 - 4} = 0$$

Hence the root of $\tan x - 4x = 0$ is 0.

### Problem

*Use Newton-Raphson method to obtain a root, correct to three decimal places of the following equations*

*1. $\sin x = 1 - x$*

*Answer: 0.511*

*2. $x^4 + x^2 - 80 = 0$*

*Answer: 2.908*

*3. $x + \log x = 2$*

*Answer: 1.756*

### Problem

*Using Newton Raphson method, find a real root correct to 3 decimal places, of the equation $\sin x = x/2$ given that the root lies between $\pi/2$ and $\pi$.*

*Answer: 1.896*

## FINITE DIFFERENCES

Assume that we have a table of values $(x_i, y_i)$ i = 0,1,2,…..n of any function y = f(x), the values of x being equally spaced, i.e., x = x + ih, i = 0,1,2,…..n. Suppose that we are required to





recover the values of f(x) for some intermediate values of f(x) for some intermediate of x, or to obtain the derivative of f(x) for some x in the range $x_o < x < x_n$.

The methods for the solution of these problems are based on the concept of the differences of a function which we now proceed to define.

## Forward Differences

If $y_0, y_1, y_2, \ldots\ldots$denote a set of values of y, then $y_1 - y_0, y_2 - y_1, \ldots\ldots y_{n-1}$ are called the **differences** of y. Denoting these differences by $\Delta y_0, \Delta y_1, \ldots\ldots\Delta y_{n-1}$ respectively.

We have

$$\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \ldots\ldots\Delta y_{n-1} = y_n - y_{n-1},$$

where **$\Delta$ is called the forward difference operator** and $\Delta y_0, \Delta y_1$ are called **first forward differences**. The differences of the first forward differences are called **second forward differences** and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots\ldots$. Similarly, one can define third forward differences, fourth forward differences, etc. Thus

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

The following table shows how the forward differences of all orders can be formed :

**Forward difference table**

| x | y | $\Delta$y | $\Delta^2$y | $\Delta^3$y |
|---|---|---|---|---|
| $x_0$ | $y_0$ | | | |
| | | $y_1 - y_0 = \Delta y_0$ | | |
| $x_1 + h$ | $y_1$ | | $\Delta y_1 - \Delta y_0 = \Delta^2 y_0$ | |
| | | $y_2 - y_1 = \Delta y_1$ | | $\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$ |
| $x_2 + 2h$ | $y_2$ | | $\Delta y_2 - \Delta y_1 = \Delta^2 y_1$ | |
| | | $\Delta y_2$ | | |
| $x_3 + 3h$ | $y_3$ | | | |

## Backward Differences

The differences $y_1 - y_0, y_2 - y_1, \ldots\ldots y_n - y_{n-1}$ are called first backward difference if they are denoted by $\nabla y_1, \nabla y_2, \ldots\ldots$respectively, so that $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \ldots\ldots\ldots\ldots \nabla y_n = y_n - y_{n-1}$ , where $\nabla$ is called the **backward difference operator**.





**Backward difference table**

| x | y | $\nabla$ | $\nabla^2$ | $\nabla^3$ |
|---|---|---|---|---|
| $x_0$ | $y_0$ | | | |
| | | $y_1 - y_0 = \nabla y_1$ | | |
| $x_1 + h$ | $y_1$ | | $\nabla y_2 - \Delta y_1 = \nabla^2 y_2$ | |
| | | $y_2 - y_1 = \nabla y_2$ | | $\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$ |
| $x_2 + 2h$ | $y_2$ | | $\nabla y_3 - \Delta y_2 = \nabla^2 y_3$ | |
| | | $\nabla y_3$ | | |
| $x_3 + 3h$ | $y_3$ | | | |

## Central Differences

The central difference operator $\delta$ is defined by the relations :

$$y_1 - y_0 = \delta y_{1/2},\ y_2 - y_1 = \delta y_{3/2}, \ldots\ldots\ldots\ldots y_n - y_{n-1} = \delta y_{2n-1/2}$$

**Central difference table**

| x | y | $\delta y$ | $\delta^2 y$ | $\delta^3 y$ |
|---|---|---|---|---|
| $x_0$ | $y_0$ | | | |
| | | $\delta_{y_{1/2}}$ | | |
| $x_1$ | $y_1$ | | $\delta^2_{y_1}$ | |
| | | $\delta_{y_{3/2}}$ | | $\delta^3_{y_{3/2}}$ |
| $x_2$ | $y_2$ | | $\delta^2_{y_2}$ | |
| | | $\delta_{y_{5/2}}$ | | |
| $x_3$ | $y_3$ | | | |





### Example

*By constructing a difference table and taking second order differences as constants, find the sixth term of the series 8, 12, 19, 29, 42, ...*

### Solution

Let us make following difference table

| x | y | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 1 | 8 | | | |
| | | 4 | | |
| 2 | 12 | | 3 | |
| | | 7 | | 0 |
| 3 | 19 | | 3 | |
| | | 10 | | 0 |
| 4 | 29 | | 3 | |
| | | 13 | | a - 58 |
| 5 | 42 | | a - 55 | |
| | | a - 42 | | |
| 6 | a | | | |

As second order differences are constants, so third order differences are all zero.

$\therefore \quad a - 58 = 0$

i.e. $\quad a = 58$

Hence sixth term is 58.

## SYMBOLIC RELATIONS

Difference formula can easily be established by symbolic methods, using the **shift operator E** and the averaging or the **mean operator μ,** in addition to the operators $\Delta$, $\nabla$ and $\delta$ already defined.

The shift operator E is defined by the equation

$$Ey_r = y_{r+1}$$

A second operation with E gives

$$E^2 y_r = E(Ey_r) = Ey_{r+1} = y_{r+2}$$

and in general,

$$E^n y_r = y_{r+n}$$

It is now easy to derive a relationship between $\Delta$ and E, for we have

$$\Delta y_0 = y_1 - y_0 = Ey_0 - y_0 = (E-1)y_0$$





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and hence $\Delta = E - 1$

or $E = 1 + \Delta$ (1)

We can now express any higher order forward difference in terms of the given function values. For Example

$$\Delta^3 y_0 = (E-1)^3 y_0$$

$$= (E^3 - 3E^2 + 3E - 1) y_0$$

$$= y_3 - 2y_2 + 3y_1 - y_0$$

Also, $\nabla = 1 - E^{-1}$ (2)

and $\delta = E^{1/2} - E^{1/2}$ (3)

$E = e^{hD}$ (4)

$\Delta = \delta E^{1/2}$ (5)

$(1 + \Delta)(1 - \nabla) = 1$ (6)

Also $\Delta[f(x)g(x)] = f(x + h)\Delta g(x) + g(x)\Delta f(x)$ (7)

Useful trigonometric formula

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

## Example

*Prove that* $\Delta^3 = E^3 - 3E^2 + 3E - 1$

## Solution

By the definition we have

$\Delta f(x) = f(x + h) - f(x)$

$E f(x) = f(x + h)$

and $E^n f(x) = f(x + nh)$ (1)

$\therefore$ $\Delta^2 f(x) = \Delta[f(x + h) - f(x)]$

$= f(x + 2h) - 2f(x + h) + f(x)$

and





$$\Delta^3 f(x) = \Delta[f(x+2h) - 2f(x+h) + f(x)]$$
$$= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$
$$= E^3 f(x) - 3E^2 f(x) + 3E f(x) - f(x) \quad \text{[Using (1)]}$$
$$\Delta^3 f(x) = (E^3 - 3E^2 + 3E - 1) f(x)$$

or $\Delta^3 \equiv E^3 - 3E^2 + 3E - 1.$

Hence proved

## Problem

*Prove that $\Delta 2 \equiv E^2 - 2E + 1$.*

## Example

$\Delta^2(3e^x)$

## Solution

$$\Delta^2[3e^x] = 3\Delta^2[e^x] = 3\Delta.\Delta[e^x] = 3\Delta[e^{x+1} - e^x]$$
$$= 3[e^{x+2} - e^{x+1} - e^{x+1} + e^x]$$
$$= 3[e^2 - 2e + 1]e^x$$
$$= 3(e-1)^2 e^x$$

## Example

*Evaluate $\Delta[e^{2x}\log 3x]$*

## Solution

We know $\Delta[f(x)g(x)] = f(x+h)\Delta g(x) + g(x)\Delta f(x)$

Given that

$f(x) = e^{2x}$

$g(x) = \log 3x$

Now $\Delta(e^{2x}\log 3x) = e^{2(x+h)}\Delta\log 3x + \log 3x \Delta e^{2x}$

$$= e^{2(x+h)}[\log 3(x+h) - \log 3x] + \log 3x(e^{2(x+h)} - e^{2x})$$
$$= e^{2x}\left[e^{2h}\log\left(1+\frac{h}{x}\right) + (e^{2h}-1)\log 3x\right]$$





### Problem

*Evaluate $\Delta (x^2 + e^x + 2)$.*

*Answer: $2hx + h^2 + e^x(e^h - 1)$.*

### Problem

*Evaluate $\Delta [\log x]$*

*Answer:* $\log\left(1+\frac{h}{2}\right)$

### Example

*Evaluate $\Delta [\text{Sin } (ax + b)]$.*

### Solution

$\Delta[\sin(ax + b)] = \sin[a(x + h) + b] - \sin(ax + b)$

Now using the formula

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\Delta[\sin(ax + b)] = \sin[a(x + h) + b] - \sin(ax + b)$$

$$= 2\cos\left(\frac{[a(x+h)+b]+[ax+b]}{2}\right)\sin\left(\frac{[a(x+h)+b]-[ax-b]}{2}\right)$$

$$= 2\cos\left(ax+b+\frac{ah}{2}\right)\sin\frac{ah}{2}$$

### Problem

*Evaluate $\Delta (x + \cos x)$*

*Answer:* $h-2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$

### Example

*Establish* $\delta = E^{1/2} - E^{-1/2}$

### Solution

$$\delta y_x = y_{x+\frac{1}{2}h} - y_{x-\frac{1}{2}h}$$





$$= E^{1/2} y_x - E^{-1/2} y_x$$

$$= (E^{1/2} - E^{-1/2}) y_x$$

$$\delta = E^{1/2} - E^{-1/2}$$

## Example

*Establish* $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$ *where* $\mu$ *is average operator.*

## Solution

$$\mu y_x = \frac{1}{2}\left( y_{x+\frac{1}{2}h} + y_{x-\frac{1}{2}h} \right)$$

$$= \frac{1}{2}\left(E^{1/2} y_x + E^{-1/2} y_x\right)$$

$$= \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) y_x$$

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

## Example (missing entry)

*Find the missing entry in the following table*

| *x* | *0* | *1* | *2* | *3* | *4* |
|---|---|---|---|---|---|
| *y(x)* | *1* | *3* | *9* | – | *81* |

## Solution

Difference table

Let $y_3 = a$

| x | y | Δy | $\Delta^2$y | $\Delta^3$y | $\Delta^4$y |
|---|---|---|---|---|---|
| 0 | 1 | | | | |
| | | 2 | | | |
| 1 | 3 | | 4 | | |
| | | 6 | | a - 19 | |
| 2 | 9 | | a - 15 | | 124 - 4a |





| | | | | |
|---|---|---|---|---|
| | | a - 9 | | 105 - 3a |
| 3 | a | | 90 - 2a | |
| | | 81 - a | | |
| 4 | 81 | | | |

Finding missing term ($y_3$)

Four entries are given

∴ 4th-order differences are zero

∴ 124 - 4a = 0

∴ a = 31

∴ y(3) = 31

## Example

*Find the missing entry in the following table*

| *x* | *0* | *5* | *10* | *15* | *20* | *25* |
|---|---|---|---|---|---|---|
| *y* | *6* | *10* | – | *17* | – | *31* |

## Solution

Difference table

| x | y | Δy | $\Delta^2$y | $\Delta^3$y | $\Delta^4$y |
|---|---|---|---|---|---|
| 0 | 6 | | | | |
| | | 4 | | | |
| 5 | 10 | | a - 14 | | |
| | | a - 10 | | 41 - 3a | |
| 10 | a | | 27 - 2a | | 6a + b - 102 |
| | | 17 - a | | 3a + b - 61 | |
| 15 | 17 | | a + b - 34 | | 143 - 4a - 4b |
| | | b - 17 | | 82 - a - 3b | |
| 20 | b | | 48 - 2b | | |
| | | 31 - b | | | |





| | |
|---|---|
| 25 | 31 |

Finding a and b

As four entries are given, so 4th-order differences are zero

$\therefore \quad 6a + b - 102 = 0 \quad (1)$

$143 - 4a - 4b = 0 \quad (2)$

$\therefore \quad 6a + b = 102 \quad (3)$

$4a + 4b = 143 \quad (4)$

Solving (3) and (4)

$a = 13.25$ and $b = 22.50$

### Problem

*Estimate the missing term in the following:*

| *x* | *1* | *2* | *3* | *4* | *5* | *6* | *7* |
|---|---|---|---|---|---|---|---|
| *y* | *2* | *4* | *8* | – | *32* | *64* | *128* |

*Answer: 16.1*

## NEWTON'S FORMULA FOR INTERPOLATION

Given the set of (n+1) values, viz, $(x_0 , y_0)$, $(x_1 , y_1)$, $(x_2 , y_2)$, ……$(x_n , y_n)$ of x and y, it is required to find $y_n(x)$.

Let the values of x be equidistant, i.e., let $x_i = x_0 + ih$, $[i = 0,1,2,\ldots\ldots n]$

Then

$$y_1(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3}\Delta^3 y_0 + \frac{p(p-1)(p-2).....(p-n+1)}{n}\Delta^n y_0$$

Which is **Newton's forward difference interpolation formula** and is useful for interpolation **near the beginning of a set of tabular values**.

$$y_n(x) = y_n + p\nabla y_n = \frac{p(p+1)}{2}\nabla^2 y_n + ..... + \frac{p(p+1).....(p+n-1)}{n}$$

Where $p = \dfrac{x - x_n}{h}$

This is **Newton's backward difference interpolation formula** and it uses tabular values to the left n of y. This formula is therefore useful for interpolation **near the end of tabular values**.





## Example (AKTU 2011)

*The population of a town decennial census was as given below. Estimate the population for the year 1895.*

| ***Year*** | ***x*** | ***1891*** | ***1901*** | ***1911*** | ***1921*** | ***1931*** |
|---|---|---|---|---|---|---|
| *Population (in thousand)* | *y* | *46* | *66* | *81* | *93* | *101* |

Finding p

Putting h = 10, $x_0$ = 1891, x = 1895 in the formula $x = x_0 + ph$

We obtain $p = \frac{2}{5} = 0.4$.

Difference table

| x | y | Δ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ |
|---|---|---|---|---|---|
| 1891 | 46 | | | | |
| | | 20 | | | |
| 1901 | 66 | | -5 | | |
| | | 15 | | 2 | |
| 1911 | 81 | | -3 | | -3 |
| | | 12 | | -1 | |
| 1921 | 93 | | -4 | | |
| | | 8 | | | |
| 1931 | 101 | | | | |

Apply Newton's forward difference interpolation formula

Hence using Newton's **forward** difference interpolation formula (because 1895 is near the beginning)

$$y(1895) = 46 + 0.4(2) + \frac{0.4(0.4-1)}{6}(-5) + \frac{0.4(0.4-1)(0.4-2)}{6}(2) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$

= 54.85 thousand

## Example

*In above Example , estimate the population for the year 1925.*





## Solution

Finding p

Here interpolation is desired at the end of the table and so we use *Newton **backward** diff.* interpolation formula (because 1925 is near the end)

$x = x_n + p\,h.$

With $x = 1925,\ x_n = 1931$ And $h = 10.$

We obtain $p = -0.6.$

Apply Newton backward diff. interpolation formula

$$y(1925) = 101 - (0.6)8 + \frac{-0.6(-0.6+1)}{4}(-4) + \frac{-0.6(-0.6+1)(-0.6+2)}{6}(-4) + \frac{-0.6(0.6+1)(-0.6+2)(-0.6+3)}{24}(-3)$$

= 96.84 thousand

## Example

*The table below gives the values of tan x for 0.10 < x < 0.30.*

| *x* | *0.10* | *0.15* | *0.20* | *0.25* | *0.30* |
|---|---|---|---|---|---|
| *y = tan x* | *0.1003* | *0.1511* | *0.2027* | *0.2553* | *0.3093* |

*Find : (i) tan 0.12 (ii) tan 0.25*

## Solution

Difference table

| x | y | Δ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ |
|---|---|---|---|---|---|
| 0.10 | 0.1003 | | | | |
| | | 0.0508 | | | |
| 0.15 | 0.1511 | | 0.0008 | | |
| | | 0.0516 | | 0.0002 | |
| 0.20 | 0.2027 | | 0.0010 | | 0.0002 |
| | | 0.0526 | | 0.0004 | |
| 0.25 | 0.2553 | | 0.0014 | | |
| | | 0.0540 | | | |
| 0.30 | 0.3093 | | | | |

Finding tan(0.12)

To find tan (0.12), we have 0.12 = 0.10 + p(0.05) which gives p = 0.40.





Hence using Newton's **forward** diff. interpolation formula :

$$\tan(0.12) = 0.1003 + 0.4\,(0.0508) + \frac{0.4(0.4-1)}{2}(0.0008) + \frac{0.4(0.4-10(0.4-2)(0.4-3)}{6}\,(0.0002) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}\,(0.0002) = 0.1205.$$

Finding tan (0.26)

To find tan (0.26), we have 0.26 = 0.30 + p (0.05) which gives p = -0.8.

Hence using Newton's **backward** diff. interpolation formula :

$$\tan(0.26) = 0.3093 + 0.8\,(0.0540) + \frac{-0.8(0.8-1)}{2}\,(0.0014) + \frac{-0.8(-0.8+1)(-0.8+2)}{6}\,(0.0004) + \frac{-0.8(-0.8+1)(-0.8+2)(-0.8+3)}{24}\,(0.0002) = 0.2662.$$

## Problem

*Using Newton's forward difference formula find the value of f(1.6) if*

| *x* | *1* | *1.4* | *1.8* | *2.2* |
|---|---|---|---|---|
| *f(x)* | *3.49* | *4.82* | *5.96* | *6.5* |

*Answer: 5.439375*

## Problem

*From the following table, estimate the number of students who obtained marks between 40 and 45.*

| *Marks* | *30–40* | *40–50* | *50–60* | *60–70* | *70–80* |
|---|---|---|---|---|---|
| *No. of students* | *31* | *42* | *51* | *35* | *31* |

*Answer: 17*

*Hint: First prepare the cumulative frequency table as follows:*

| *Marks (less than)* | *40* | *50* | *60* | *70* | *80* |
|---|---|---|---|---|---|





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| | | | | | |
|---|---|---|---|---|---|
| *No. of students* | *31* | *42* | *51* | *35* | *31* |

*Now proceed as usual.*

## LAGRANGE'S INTERPOLATION FORMULA FOR UNEQUAL INTERVALS

$$y_x = \frac{(x-x_2)(x-x_3)....(x-x_n)}{(x_1-x_2)(x_1-x_3)....(x_1-x_n)}y_1 + \frac{(x-x_1)(x-x_3)....(x-x_n)}{(x_2-x_1)(x_2-x_3)....(x_2-x_n)}y_2$$
$$+ ..... + \frac{(x-x_1)(x-x_2)....(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)....(x_n-x_{n-1})}y_n$$

This method can also be used for unequal interval.

### Proof

Let $y_0$, $y_1$, $y_2$, ..., $y_n$ be the values of function y = f (x) corresponding to the arguments $x_0$, $x_1$, $x_2$, ...., $x_n$ not necessarily equally spaced. Since there are (n + 1) values of f (x) so $(n + 1)_{th}$ difference is zero. Thus f (x) is supposed to be polynomial in x of degree n.

Then $y = f(x) = a_0 (x - x_1) (x - x_2) ..... (x - x_n)$

$+ a_1 (x - x_0) (x - x_2) ....(x - x_n)$

$+ a_2(x - x_0) (x - x_1) ..... (x - x_n) .....$

$+ a_n (x - x_0) (x - x_1) ..... (x - x_n - 1)$ (1)

where $a_0$, $a_1$, $a_2$, ....., an are constants.

To determine a0 put $x = x_0$ and $y = y_0$ in (1), we get

$$y_0 = a_0(x_0 - x_1) (x_0 - x_2) ..... (x_0 - x_n)$$

It implies $$a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)}$$

To determine $a_1$, put $x = x_1$ and $y = y_1$ in (1)

$$a_1 = \frac{y_0}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)}$$

Proceeding, we get

$$a_n = \frac{y_n}{(x_n-x_1)(x_n-x_1)...(x_n-x_{n-1})}$$

Putting these values of $a_1$, $a_2$, ....., $a_n$ in (1), we get





$$y_x = \frac{(x-x_2)(x-x_3)....(x-x_n)}{(x_1-x_2)(x_1-x_3)....(x_1-x_n)}y_1 + \frac{(x-x_1)(x-x_3)....(x-x_n)}{(x_2-x_1)(x_2-x_3)....(x_2-x_n)}y_2$$
$$+ ..... + \frac{(x-x_1)(x-x_2)....(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)....(x_n-x_{n-1})}y_n$$

## Example

*Use Lagrange's interpolation formula and find the value of y at x = 3.5, with the following data for x and y.*

| ***x*** | *1* | *3* | *5* |
|---|---|---|---|
| ***y*** | *1.5708* | *1.5719* | *1.5738* |

## Solution

Here x = 3.5

The Lagrange's formula gives

$$y_{3.5} = \frac{0.5(-1.5)}{-2(-4)}(1.5708) + \frac{2.5(-1.5)}{2(-2)}(1.5719) + \frac{2.5x0.5}{4x2}(1.5738) = 1.5723$$

## Problem

*Given the values*

| *x* | *5* | *7* | *11* | *13* | *17* |
|---|---|---|---|---|---|
| *f(x)* | *150* | *392* | *1452* | *2366* | *5202* |

*Evaluate f(9) by using Lagrange's formula.*

*Answer: 810*

## Problem

*Find the interpolating polynomial for (0, 2), (1,3), (2,12) and (5,147) using Lagrange's interpolation formula.*

## Solution

Form the table

| x | 0 | 1 | 2 | 5 |
|---|---|---|---|---|
| y | 2 | 3 | 12 | 147 |

Apply Lagrange's interpolation formula





If P(x) is Lagrange interpolating polynomial for y(x) then

$$P(x)=\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-5)}(2)+\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(2)$$
$$+\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12)+\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$= \frac{1}{5}\left(x^3-8x^2+17x-10\right)+\frac{3}{4}\left(x^3-7x^2+10x\right)$$
$$-2\left(x^3-6x^2+5x\right)+\frac{49}{20}\left(x^3-3x^2+2x\right)$$

$= x^3 + x^2 - x + 2$

### Example

*Obtain a unique polynomial f (x) of degree 2 or less such that f (1) = 1, f (3) = 27, f (4) = 64 using Lagrange's interpolation concept.*

*Answer: $f(x) = 8x^2 - 19x + 12$*

## NEWTON'S GENERAL INTERPOLATION FORMULA (DIVIDED DIFFERENCES)

Let $(x_0, y_0)$, $(x_1, y_1)$,……..,$(x_n, y_n)$ be the given (n + 1) points.

$$y = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2]$$
$$+(x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3]+.....$$
$$+(x-x_0)(x-x_1).....(x-x_n)[x,x_0,x_1,.....x_n]$$

This formula is called Newton's general interpolation formula with **divided differences**, the last term being the reminder term after (n+1) terms.

### Example

*As our example to illustrate the use of Newton's divided difference formula, we consider the data given below :*

The divided difference table is

| x | $\log_{10}x$ | $\Delta f(x)$ | $\Delta^2 f(x)$ |
|---|---|---|---|
| 300 | 2.4771 | | |
| | | 0.00145 | |
| 304 | 2.4829 | | 0.00001 |
| | | 0.00140 | |
| 305 | 2.4843 | | 0 |

| | | | |
|---|---|---|---|
| | | | 0.00140 |
| 307 | 2.4871 | | |

Hence equation of divided difference gives

$$\log_{10}301 = 2.4771 + 0.00145 + (-3)\,(-\,0.00001) = 2.4786.$$

*It is clear that the arithmetic in this method is much simpler when compared to that in Lagrange's method.*

## Example

Using the following table find f(x) as a polynomial in x from divided difference method.

| **x** | -1 | 0 | 3 | 6 | 7 |
|---|---|---|---|---|---|
| **f(x)** | 3 | -6 | 39 | 822 | 1611 |

The difference table is

| x | f(x) | Δf(x) | $\Delta^2$f(x) | $\Delta^3$f(x) | $\Delta^4$f(x) |
|---|---|---|---|---|---|
| -1 | 3 | | | | |
| | | -9 | | | |
| 0 | -6 | | 6 | | |
| | | 15 | | 5 | |
| 3 | 39 | | 41 | | 1 |
| | | 261 | | 13 | |
| 6 | 822 | | 132 | | |
| | | 789 | | | |
| 7 | 1611 | | | | |

Hence equation of divided difference gives

$$f(x) = 3 + (x+1)(-9) + x(x+1)(6) + x(x+1)(x-3)(5) + x(x+1)(x-3)(x-6)$$

$$= x^4 - 3x^3 + 5x^2 - 6$$

## Problem

*Given the values*

| *x* | *5* | *7* | *11* | *13* | *17* |
|---|---|---|---|---|---|
| *f(x)* | *150* | *392* | *1452* | *2366* | *5202* |

*Evaluate f(9) by using Newton's divided difference formula.*

*Answer: 810*





## ASSIGNMENT

**Q.1. (JNTUH 2018, 2 marks):** Give the graphical interpretation of the bisection method.

**Q.2. (PTU NM 2018, 2 marks):** Write a short note on Bisection method.

**Q.3. (PTU NM 2018, 2 marks):** Define transcendental equation.

Answer: A polynomial equation is an equation in the form of $x^4 - 4x^2 - 3 = 0$, $4x^2 - 3x + 9 = 0$ are are some of the algebraic equations.

An equation containing polynomials, logarithmic functions, trigonometric functions, and exponential functions is known as transcendental equation. $\tan x - e^x = 0$, $\sin x - xe^{2x} = 0$, and $xe^x = \cos x$ are some of the transcendental equations examples

**Q.4. (AKTU 2018, 2020, 2 marks):** Give comparison between Regula-Falsi method and Newton Raphson method.

**Q.5. (GTU NM 2020, 3 marks):** State the formulae for Newton's backward interpolation methods . Specify the methods used for unequal intervals

**Q.6. (AKTU 2018, 2020, 2021, 2 marks):** Write the relation between nth divided difference and nth forward difference.

**Q.7. (AKTU 2021, 7 marks):** Write the order of convergence of an iterative method. Prove that rate of convergence of Newton Raphson method is quadratic.

**Q.8. (BPUT 2020, 2 marks):** Write the advantages of newton divided difference method over langrage's method.

Answer: The advantage of Newton interpolation is the use of nested multiplication and the relative easiness to add more data points for higher-order interpolating polynomials.

**Q.9. (UTU 2013):** Determine the value of p and q so that rate of convergence of the iterative method

$$x_{n+1} = +px_n + q\frac{N}{x_n^2}$$

for computing $N^{1/3}$ becomes as high as possible.

Answer: p = 2/3; q = 1/3

**Q.10. (JNTUK NM 2020, 7 marks):** Find the root of the equation $2x - \log x_{10} = 7$ using Iteration method.

Answer: 3.7893

**Q.11. (JNTUH 2018, 2 marks):** Write the iterative formula for finding $\sqrt[3]{N}$ using method of false position.

**Q.12. (JNTUH 2019, 2 marks):** Write the iterative formula for finding $\sqrt{N}$ using Newton Raphson method.

Answer: Newton's Iterative Formula to Find $b_{th}$ Root of a Positive Real Number N is

$$x_{n+1} = \frac{1}{b}\left[(b-1)x_n + \frac{N}{x_n^{b-1}}\right]$$

**Q.13. (KU 2019, 7.5 marks):** Find the real root of the equation $x^3 - 4x - 9 = 0$ using Bisection method in four stages.

Answer: 2.6875

**Q.14. (JNTUK NM 2020, 7 marks):** Find the root of the equation $3x = 1 + \cos x$ using bisection method.

Answer:

**Q.15. (AKTU 2018, 2020, UTU 2013, 7 marks):** State and prove the Lagrange interpolation formula. Find the interpolating polynomial by Lagrange interpolation formula for the given data

| x | 5 | 6 | 9 | 11 |
|---|---|---|---|---|
| y | 12 | 13 | 14 | 16 |





**Q.16. (GTU NM 2020, 4 marks):** Find the value of f(5) by using Lagrange's interpolation method for the given data.

| x | 1 | 2 | 3 | 4 | 7 |
|---|---|---|---|---|---|
| f(x) | 2 | 4 | 8 | 16 | 128 |

Answer: 31.27

**Q.17. (GTU NM 2020, 4 marks):** Find the value of f(3) by using Lagrange's interpolation method for the given data.

| x | 0 | 1 | 2 | 5 |
|---|---|---|---|---|
| f(x) | 2 | 3 | 12 | 147 |

Answer: 35

**Q.18. (AKTU 2019, 3.5 marks):** Find the unique polynomial $P(x)$ of degree 2 such that: $P(1) = 1$, $P(3) = 27$, $P(4) = 64$, use Lagrange method of interpolation.

Answer: $8x^2 - 19x + 12$.

**Q.19. (AKTU 2019, 3.5 marks):** Solve $x^3 - 5x + 3 = 0$ by using Regula-Falsi method correct up to four decimal places.

Answer: 0.6566

**Q.20. (JNTUH 2019, 5 marks):** Find the root of the equation $2x - \log x = 7$ which lies between 3.5 and 4 by Regula-Falsi method.

Answer: 3.789

**Q.21. (UTU 2015, JNTUK NM 2021, 7 marks):** Solve $x\log_{10}x = 1.2$ by False position method.

Answer: 2.7406

**Q.22. (UTU 2016):** Find the real root of the equation $\cos x - x.e^x = 0$ correct to four decimal place by using Regula Falsi method.

Answer: 0.5177

**Q.23. (UTU 2013):** Apply false position method to find positive root of the equation $x - e^{-x} = 0$ correct to three decimal places.

Answer: 0.567

**Q.24. (UTU 2014):** Find the root of the equation $xe^x = \cos x$ is the interval (0, 1) using Regula-Falsi method correct upto four decimal places.

Answer: 0.5177

**Q.25. (JNTU 2016, 2019, BPUT 2020, 6 marks):** Find the real root of the equation $\cos x - x.e^x = 0$ correct to three decimal place by using Newton Raphson Method.

Answer: 0.5177

**Q.26. (GTU NM 2020, 7 marks):** Describe the Newton Raphson method in brief and solve $e^x = 5x$

Answer: $x = 2.5425$

**Q.27. (JNTUK NM 2020, 2 marks):** Find the $\sqrt{17}$ using Newton Raphson method.

Answer: 4.123

Hint: $f(x) = x^2 - 17$

**Q.28. (UTU 2014):** By Newton-Raphson method, find the value of $(48)^{1/3}$.

Answer: 3.6342

**Q.29. (AKTU 2019, 3.5 marks):** From the table, estimate the number of students who obtained marks between 40 and 45.

| Marks | 30-40 | 40-50 | 50-60 | 60-70 | 70-80 |
|---|---|---|---|---|---|
| No. of students | 31 | 42 | 51 | 35 | 31 |





Answer: 47.8671875, i.e. 48.

Use Newton's forward difference interpolation formula.

**Q.30. (BPUT 2020, 6 marks):** From the following data estimate number of students who secured marks not more than 45, by using any suitable interpolation formulae.

| Marks | 30-40 | 40-50 | 50-60 | 60-70 | 70-80 |
|---|---|---|---|---|---|
| No. of students | 35 | 48 | 70 | 40 | 22 |

**Q.31. (BPUT 2020, 6 marks):** The Area of a circle of diameter d is given for the following rules.

| D | 80 | 85 | 90 | 95 | 100 |
|---|---|---|---|---|---|
| A | 5026 | 5674 | 6362 | 7088 | 7854 |

Find A for 105.

Answer: 8666

**Q.32. (BPUT 2020, 6 marks):** Using Newton's divided difference formula, calculate the value of f(6) from the following data:

| x | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| f(x) | 3 | 5 | 8 | 10 | 13 | 16 |

**Q.33. (KU 2019, 7.5 marks):** Evaluate f(9) using Newton's divided difference formula from the table given as

| x | 5 | 7 | 11 | 13 | 17 |
|---|---|---|---|---|---|
| f(x) | 150 | 392 | 1452 | 2366 | 5202 |

**Q.34. (KU 2019, 7.5 marks):** Estimate the value of f(42) from the following data:

| x | 20 | 25 | 30 | 35 | 40 | 45 |
|---|---|---|---|---|---|---|
| f(x) | 354 | 332 | 291 | 260 | 231 | 204 |

Answer: 219.143, use Newton's backward interpolation formula.

**Q.35. (JNTUK 2020, 7 marks):** Given that $Sin45^0 = 0.7071$, $Sin50^0 = 0.766$, $Sin55^0 = 0.8192$, $Sin60^0 = 0.866$ find $Sin52^0$ using Newton's Back ward difference formula.

Answer: 0.788

**Q.36. (JNTUK 2020, 7 marks):** Using Newton's backward difference formula find y(12) from the following table.

| x | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| f(x) | 7 | 11 | 14 | 18 | 24 | 32 |

**Q.37. (KU 2019, 7.5 marks):** Estimate the missing terms from the following table

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| f(x) | 5 | 11 | 22 | 40 | – | 140 | – |

**Q.38. (JNTUK NM 2021, PTU 2018, 7 marks):** Find y(1.1) using Newton Forward difference formula from the table

| X | 0 | 1.2 | 1.4 | 1.6 |
|---|---|---|---|---|
| Y | 3.49 | 4.82 | 5.91 | 6.5 |

**Q.39. (PTU NM 2018, 2 marks):** Use the forward-difference formula to approximate the derivative of f(x) = In x at $x_0 = 1.8$ using h = 0.01.

Answer: 0.555





**Q.40. (UTU 2013):** Estimate the production of cotton in the year 1935 from the data given below:

| Year (x) | 1931 | 1932 | 1933 | 1934 | 1935 | 1936 | 1937 |
|---|---|---|---|---|---|---|---|
| Production | 17.1 | 13 | 14 | 9.6 | – | 12.4 | 18.2 |

Answer: 6.606667

**Q.41. (UTU 2014):** Find the missing term in the following table:

| x | 45 | 50 | 55 | 60 | 65 |
|---|---|---|---|---|---|
| y | 3 | - | 2 | - | -2.4 |

Answer: f(50) = 2.925; f(60) = 0.225

**Q.42. (UTU 2017):** Find the missing term in the following table:

| x | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| y | 6 | - | 10 | 20 | - | 15 | 5 |

**Q.43. (UTU 2016):** Prove that $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

**Q.44. (UTU 2014):** The following table is given

| x | 0 | 1 | 2 | 5 |
|---|---|---|---|---|
| y | 2 | 3 | 12 | 147 |

What is the form of the function?

Answer: $f(x) = x^3 + x^2 - x + 2$. Use Newton's divided difference formula.

**Q.45. (UTU 2017):** Using Newton's divided difference formula find the value of f(8) and f(12).

| x | 4 | 5 | 7 | 10 | 11 | 13 |
|---|---|---|---|---|---|---|
| f(x) | 48 | 100 | 294 | 900 | 1210 | 2028 |

Answer: f (8) = 448, f (15) = 3150

**Q.46. (UTU 2017):** Find the interpolating polynomial which satisfies the following data and hence evaluate f(9.5) from the given table.

| x | 7 | 8 | 9 | 10 |
|---|---|---|---|---|
| f(x) | 3 | 1 | 1 | 9 |



